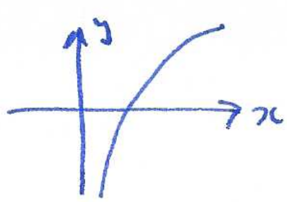
Example $y = \ln(x)$, at $x = 1$



$f(x) = \ln(x)$ $\quad f(1) = 0$

$f'(x) = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $\quad f^{(3)}(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(1) = 3!$

$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,x^{-k}$ $\quad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

so $T_n(x) = 0 + 1(x-1) + (-1)\frac{(x-1)^2}{2!} + \frac{2}{3!}(x-1)^3 + \cdots \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$

Taylor series is $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

<u>warning</u>: this may not converge! $\qquad$ alt form

$\ln(3) = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \cdots$ $\qquad \ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

<u>Q</u>: how good are the Taylor approximations?

<u>Thm</u> (error bound) $f(x)$ function, $f^{(n)}(x)$ exists and is cts.

Let K be an upper bound for $|f^{(n+1)}(u)|$ for all $u \in [a, x]$.

Then $|T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$

<u>Example</u> $f(x) = e^x$, find $T_4(x)$ at $x = 0$, find an error bound for $T_4(1)$

$f(x) = e^x$ $\quad f(0) = 1$ $\qquad T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$\vdots \qquad \vdots$

$f^{(4)}(x) = e^x$ $\quad f^{(4)}(0) = 1$ $\qquad T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.708\overline{3}$

<u>error bound</u>: need to find K s.t. $|f^{(n+1)}(x)| \leq K$ for all $x \in [0,1]$.

i.e. want $|e^u| \leq K$ for $u \in [0,1]$, can choose $K = 3$.

so $|e - T_4(1)| \leq \frac{3 \cdot 1^{n+1}}{(n+1)!} = \frac{3}{120}$ $\qquad$ [quick upper bound for e:
$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$
$\leq 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3$]