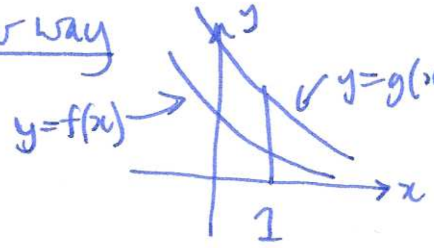
<u>Example</u> $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ note: $\sqrt{x^3+1} \geq \sqrt{x^3}$ on $[1,\infty)$

so $0 \leq \frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}}$ on $[1,\infty)$

<u>try</u>: $\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \int_1^\infty x^{-3/2}\,dx = \lim_{R\to\infty} \int_1^{R} x^{-3/2}\,dx = \lim_{R\to\infty} \left[-2x^{-1/2}\right]_1^R$

$= \lim_{R\to\infty} \frac{-2}{\sqrt{R}} + 2 = 2 < \infty$. converges $\Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges.
(but don't know exact value)

<u>other way</u>



$0 \leq f(x) \leq g(x)$ on $[1,\infty)$

if $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow \int_1^\infty g(x)$ diverges.

<u>note</u>: $\int_1^\infty g(x)\,dx$ diverges $\not\Rightarrow \int_1^\infty f(x)\,dx$ diverges

$\int_1^\infty f(x)\,dx$ converges $\not\Rightarrow \int_1^\infty g(x)\,dx$ converges.

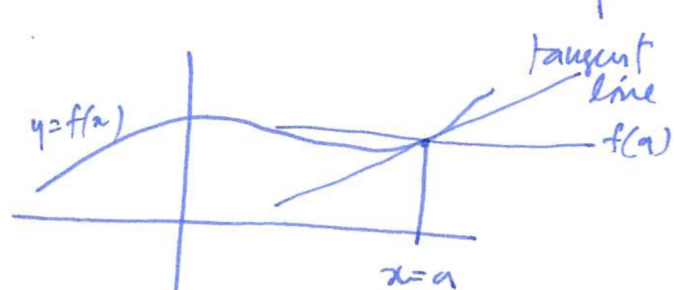
<u>Example</u> show $\int_2^\infty \frac{1}{x-\sqrt{x}}\,dx$ diverges. $x-\sqrt{x} < x$

$\frac{1}{x-\sqrt{x}} > \frac{1}{x}$

$\int_2^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_2^R = \lim_{R\to\infty} \ln|R| - \ln|2| \to \infty$.
diverges.

<u>§8.4 Taylor polynomials</u>

Approximating functions at $x=a$.



0-th approx: $f(x) \approx f(a)$

1-st order approx: (straight line) $f(x) \approx f(a) + f'(a)(x-a)$

2nd order approx: (quadratic)

3rd order approx: (cubic) ? how do we find these?

tangent line
$y - y_0 = m(x-x_0)$
$y - f(a) = f'(a)(x-a)$

tangent line: (unique) line with: same value ~~at as f(a)~~ at $x=a$ ($f(a)$).
same slope as at $x=a$ ($f'(a)$).

quadratic : same value at $x=a$: $f(a)$
same slope at $x=a$: $f'(a)$
same 2nd derivative at $x=a$: $f''(a)$
} i.e. find quadratic ax^2+bx+c which satisfies this

cubic : same value and first three derivatives, up to $f'''(a)$.

etc.

quadratic $\quad T_2(x) = ax^2+bx+c$

better $\quad T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$

deg 0 : $T_2(a) = a_0 = f(a)$ (same value)

deg 1: $T_2'(x) = a_1 + 2a_2(x-a)$

$T_2'(a) = a_1 = f'(a)$ (same first derivative)

deg 2: $T_2''(x) = 2a_2$

$T_2''(a) = 2a_2 = f''(a)$ (same second derivative)

so $\quad T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

In general $\quad T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

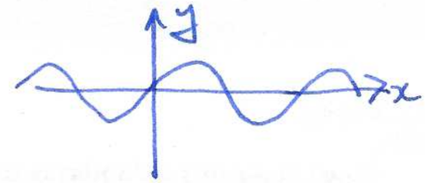
differentiate k times : $T_n^{(k)}(x) = k!a_k +$ (stuff with $(x-a)$ factors)

$T_n^{(k)}(a) = k!a_k = f^{(k)}(a)$ so $a_k = \frac{1}{k!}f^{(k)}(a)$

so $\quad T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example ① $y = \sin(x)$ at $x = 0$



$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

so $T_4(x) = 0 + 1.x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4$

$= x - \frac{x^3}{3!}$

② $f(x) = \cos(x)$ at $x = 0$

$f(x) = \cos(x) \qquad f(0) = 1$

$f'(x) = -\sin(x) \qquad f'(0) = 0$

$f''(x) = -\cos(x) \qquad f''(0) = -1$

$f^{(3)}(x) = \sin(x) \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$

so $T_4(x) = 1 + 0.x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

③ $y = e^x$ at $x = 0$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

__sneak preview__: what about infinite sums? $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

↑ __not__ a polynomial

called a __Taylor series__.

__Fact__: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all $x \in \mathbb{R}$. $\mathbb{C}$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$

__Remark__ $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$

$i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$e^{i\pi} = -1!$