§7.6 Strategies for integration

<u>tools</u>: rewrite expressions, e.g: $\dfrac{x^3-1}{x-1} = \dfrac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1$

- substitution
- parts
- trig integrals
- partial fractions

$\dfrac{x-x^3}{\sqrt{x}} = x^{1/2} - x^{5/2}$

<u>Examples</u> ① $\int x^3\sqrt{1+x^2}\,dx$ try: $u = 1+x^2$, $\dfrac{du}{dx} = 2x$ $\quad \int x^3\sqrt{u}\,\dfrac{dx}{du}\,du$

$= \int x^3\sqrt{u}\,\frac{1}{2x}\,du = \frac{1}{2}\int x^2\sqrt{u}\,du = \frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du$

$= \frac{1}{2}\,\frac{2}{5}\,u^{5/2} - \frac{1}{2}\,\frac{2}{3}\,u^{3/2} + c = \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + c$

② $\int \dfrac{1}{\sqrt{\sqrt{x}+1}}\,dx$ try: $\cancel{u=\sqrt{x}}$ $u = \sqrt{x}+1$, $\dfrac{du}{dx} = \frac{1}{2}x^{-1/2}$ $\quad \int \dfrac{1}{\sqrt{u}}\cdot\dfrac{1}{\frac{1}{2}\frac{1}{\sqrt{x}}}\,du = 2\int \dfrac{u-1}{\sqrt{u}}\,du$

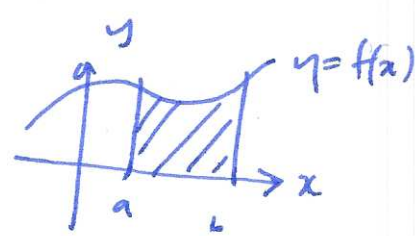
$= 2\int u^{1/2} - u^{-1/2}\,du = 2\cdot\frac{2}{3}u^{3/2} - 2u^{1/2} + c = \cancel{\frac{4}{3}(x+1)^{3/4} - 2(x+1)^{1/4} + c}$

$= \frac{4}{3}(\sqrt{x}+1)^{3/2} - 2(\sqrt{x}+1)^{1/2} + c$

③ $\int \sqrt{x^2+2x+2}\,dx$ complete the square $\int \sqrt{(x+1)^2+1}\,dx$ trig sub...

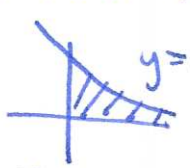
§7.7 Improper integrals

recall $\int_a^b f(x)\,dx$ = area under the curve



<u>Q</u>: what about integrals over infinite intervals?

<u>Example</u>

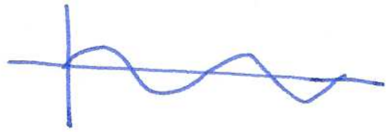


$\int_0^\infty e^{-x}\,dx$ <u>note</u>: $\int_0^R e^{-x}\,dx = \left[-e^{-x}\right]_0^R = -e^{-R} + e^0 = 1 - e^{-R}$

<u>Defn</u>: $\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx$ if this limit exists, otherwise DNE / undefined

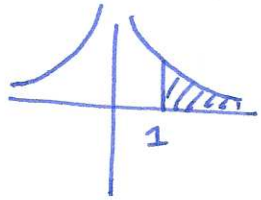
<u>warning</u>: sometimes the limit doesn't exist

<u>Example</u> $\int_0^\infty \sin(x)\,dx$



$$\int_0^R \sin(x)\,dx = \left[-\cos(x)\right]_0^R = 1-\cos(R) \qquad \lim_{R\to\infty} 1-\cos(R) \text{ DNE.}$$

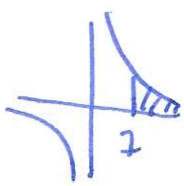
<u>Examples</u> ① $\int_1^\infty \frac{1}{x^2}\,dx$



$$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = -\frac{1}{R}+1$$

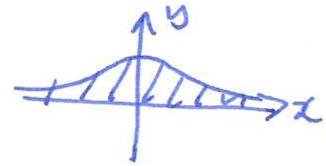
$$\lim_{R\to\infty} 1-\frac{1}{R} = 1$$

② $\int_1^\infty \frac{1}{x}\,dx$



$$\lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \left[\ln|x|\right]_1^R = \ln|R| \to \infty \text{ as } R\to\infty$$

<u>Doubly infinite integrals</u> $\int_{-\infty}^\infty f(x)\,dx$ (f continuous!)



<u>Defn</u> (f cts) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$, provided each limit exists

<u>Example</u>

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to\infty} \int_{-R}^0 \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_{-R}^0 + \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_0^R$$

$$= \lim_{R\to\infty} 0+\tan^{-1}(R) + \lim_{R\to\infty} \tan^{-1}(R) - 0 = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

<u>warning</u> $\int_{-\infty}^\infty f(x)\,dx \neq \lim_{R\to\infty} \int_{-R}^R f(x)\,dx$

<u>Example</u> $\int_{-\infty}^\infty \sin(x)\,dx$ DNE but $\lim_{R\to\infty} \int_{-R}^R \sin(x)\,dx = \lim_{R\to\infty} \left[-\cos(x)\right]_{-R}^R = \lim_{R\to\infty} 0 = 0$

Example $\int_0^\infty x e^{-x}\, dx = \lim_{R\to\infty} \int_0^R x e^{-x}\, dx$

$\int uv'\, dx = uv - \int u'v\, dx$

$u = x \quad v' = e^{-x}$

$u' = 1 \quad v = -e^{-x}$

$= \lim_{R\to\infty} \left[-xe^{-x}\right]_0^R - \lim_{R\to\infty} \int_0^R e^{-R}\, dx = \lim_{R\to\infty} \underbrace{-Re^{-R}}_{\to 0} + \lim_{R\to\infty} \left[-e^{-R}\right]_\infty^R$

$= \lim_{R\to\infty} -e^{-R} + 1 = 1$

Example when does $\int_1^\infty \frac{1}{x^p}\, dx$ converge?

$p = 1$ no

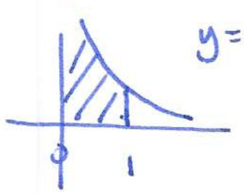
$p = 2$ yes.

$\lim_{R\to\infty} \int_1^R \frac{1}{x^p}\, dx = \lim_{R\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^R = \lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}$

$\lim R^{-p+1} = 0$ if $p > 1$

$= \infty$ if $p < 1$

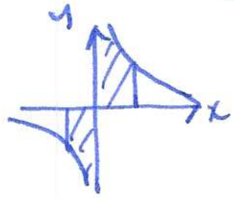
Integrals with discontinuities



$y = \frac{1}{\sqrt{x}}$ $\quad \int_0^1 \frac{1}{\sqrt{x}}\, dx$ is an improper integral! $f(0)$ not defined.

$= \lim_{R\to 0} \int_R^1 \frac{1}{\sqrt{x}}\, dx = \lim_{R\to 0} \left[2x^{1/2}\right]_R^1 = \lim_{R\to 0} 2 - 2\sqrt{R} = 2$

Warning: $\int_{-1}^1 \frac{1}{x}\, dx \neq \left[\ln|x|\right]_{-1}^1 = \ln|1| - \ln|-1| = 0$ X wrong!

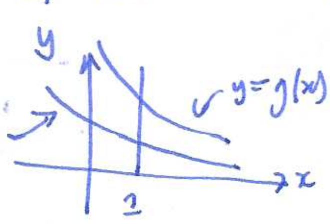


$[-1,1]$ contains a discontinuity for $\frac{1}{x}$, so need to evaluate

$\int_{-1}^0 \frac{1}{x}\, dx + \int_0^1 \frac{1}{x}\, dx = \lim_{R\to 0} \int_{-1}^R \frac{1}{x}\, dx + \lim_{R\to 0} \int_R^1 \frac{1}{x}\, dx$

$= \lim_{R\to 0} \left[\ln|x|\right]_{-1}^R + \lim_{R\to 0} \left[\ln|x|\right]_R^1 = \lim_{R\to 0} \ln|R| - 0$ DNE.

Comparison test



suppose $0 \le f(x) \le g(x)$ on $[1, \infty)$

if $\int_1^\infty g(x)\, dx$ converges then $\int_1^\infty f(x)$ converges.