Special case: quadratic factors

$x^2+1 = (x-i)(x+i) \neq (x+a_1)(x+a_2)$ a_i real.

complex roots

note: $\int \frac{1}{1+x^2}dx = \tan^{-1}(x)+c$

Q: $\int \frac{1}{1+x+x^2}dx$? A: complete the square: $x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$

$= \int \frac{1}{\frac{3}{4}+(x+\frac{1}{2})^2}dx$ sub $u = x+\frac{1}{2}$ $\frac{du}{dx}=1$ $\int \frac{1}{\frac{3}{4}+u^2}du = \frac{4}{3}\int \frac{1}{1+(\frac{2u}{\sqrt{3}})^2}du$

sub $v = \frac{2u}{\sqrt{3}}$ $\frac{dv}{du} = \frac{2}{\sqrt{3}}$ $\rightsquigarrow$ $\frac{4}{3}\int \frac{1}{1+v^2}\cdot\frac{\sqrt{3}}{2}dv = \frac{2}{\sqrt{3}}\tan^{-1}(v)+c$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right)+c$

Linear and quadratic factors

$\int \frac{1}{x(1+x^2)}dx$ $\frac{1}{x(1+x^2)} = \frac{A}{x}+\frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(1+x^2)}$

$0x^2+0x+1 = x^2(A+B)+x(C)+A$ $\left.\begin{array}{l}A+B=0\\C=0\\A=1\end{array}\right\}$ $B=-1$

$\int \frac{1}{x}+\frac{-x}{x^2+1}dx = \ln|x|-\frac{1}{2}\ln|x^2+1|+c$

Repeated quadratic factors

$\frac{1}{(x+a)(x+b)^2(x^2+c^2)^2} = \frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{(x+b)^2}+\frac{Dx+E}{x^2+c}+\frac{Fx+G}{(x^2+c)^2}$

solve for $A \dots G$.

moral: every $\frac{P(x)}{Q(x)}$ can be integrated $\square$.