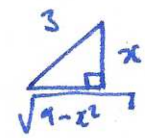
Example $\int \sqrt{9-x^2}\, dx$ try: $x = 3\sin u \Leftrightarrow u = \sin^{-1}\left(\frac{x}{3}\right)$ $\cos 2u = \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$

$\frac{dx}{du} = 3\cos u$

$$\int \sqrt{9 - 9\sin^2 u}\, \frac{dx}{du}\, du = \int 3\sqrt{1-\sin^2 u}\; 3\cos u\, du = \int 9\sqrt{\cos^2 u}\, du \cos u\, du$$

$$= 9\int \cos^2 u\, du = \frac{9}{2}\int \cos 2u + 1\, du = \frac{9}{4}\sin 2u + \frac{9}{2}u + c$$



$\frac{x}{3} = \sin u$

$$= \frac{9}{4}\cdot 2\sin u\cos u + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c = \frac{9}{2}x\sqrt{1-\left(\frac{x}{3}\right)^2} + \frac{9}{2}\sin^{-1}(u) + c$$

Example $\int \sqrt{1+4x^2}\, dx$ try: $x = \frac{1}{2}\tan u$

$\frac{dx}{du} = \frac{1}{2}\sec^2 u$

$$\int \sqrt{1 + 4\left(\tfrac{1}{2}\tan u\right)^2}\, \frac{dx}{du}\, du = \int \sqrt{1+\tan^2 u}\, \frac{1}{2}\sec^2 u\, du = \frac{1}{2}\int \sec^3 u\, du$$

$$= \frac{1}{2}\int \underbrace{\sec u}_{u} \cdot \underbrace{\sec^2 u}_{v'}\, du$$

$u = \sec u$ $v' = \sec^2 u$

$u' = \sec u \tan u$ $v = \tan u$

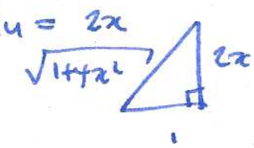
$$= \frac{1}{2}\sec u\tan u - \frac{1}{2}\int \sec u \underbrace{\tan^2 u}_{= \sec^2 u - 1}\, du$$

$$\frac{1}{2}\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\int \sec u\, du - \frac{1}{2}\int \sec^2 u\, du$$

$$\int \sec^3 u\, du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + c$$

$$\frac{1}{2}\int \sec^3 u\, du = \frac{1}{4}\sec u\tan u + \frac{1}{4}\ln|\sec u + \tan u| + c$$

$\tan u = 2x$



$$= \frac{1}{4}\; \frac{1}{\sqrt{1+4x^2}}\, 2x + \frac{1}{4}\ln|\sec u + \tan x| + c$$

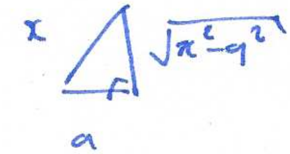
Example $\int \frac{1}{x^2\sqrt{x^2-a^2}}\, dx$ try: $x = a\sec u$

$\frac{dx}{du} = a\sec u\tan u$

$$\int \frac{1}{a^2\sec^2 u}\,\frac{1}{\sqrt{a^2\sec^2 u - a^2}}\,\frac{dx}{du}\,du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\,\frac{1}{\tan u}\,a\sec u\tan u\,du$$

$$= \frac{1}{a^2}\int \cos u\,du = \frac{1}{a^2}\sin u + c = \frac{1}{a^2}\,\frac{\sqrt{a^2-x^2}}{a} + c$$

$\sec u = \frac{x}{a}$



$$= \frac{1}{a^2}\sqrt{1 - x^2/a^2} + c$$

§7.6 Partial fractions

aim: evaluate $\int \frac{P(x)}{Q(x)}\,dx$ $\quad P(x), Q(x)$ polynomials

recall: $\int \frac{1}{x+a}\,dx = \ln|x+a| + c$

special case: $\deg P < \deg Q$ and $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i all distinct real numbers.

Fact: then $\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \dots + \frac{A_n}{x+a_n}$ $\quad A_i$ numbers.

Example $\int \frac{x}{x^2-1}\,dx$ $\qquad x^2-1 = (x+1)(x-1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x-1)(x+1)} = \frac{x(A+B)+(B-A)}{x^2-1}$$

$$\left.\begin{array}{ll} A+B=1 & (1) \\ -A+B=0 & (2) \end{array}\right\} \quad (1)+(2) \quad 2B=1 \quad B=\tfrac{1}{2},\ A=\tfrac{1}{2}.$$

so $\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$

short cut: $x = A(x-1) + B(x+1)$ ← needs to hold for all x.

in particular, for $x=1$: $\quad 1 = 2B \quad B=\frac{1}{2}$

$x=-1$: $\quad -1 = -2A \quad A = \frac{1}{2}$

special case: as above, but $\deg P \geq \deg Q$, do long division

Example $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

$$\begin{array}{r} x \\ x^2-1 \,\big)\!\overline{\;x^3\;} \\ x^3 - x \\ \hline x \end{array} \leftarrow \text{remainder}$$

so $x^3 = (x^2-1)x + x$

so $\displaystyle\int \frac{x^3}{x^2-1}\,dx = \int \frac{x}{x^2-1} + x\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1} + x\,dx$

$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{2}x^2 + c$

special case: $\dfrac{P(x)}{Q(x)}$ $\quad Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i not all distinct, i.e. repeated roots.

Example $\displaystyle\frac{x+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ $\quad A, B, C$ real numbers.

in general: $\dfrac{P(x)}{Q(x)}$ $\quad Q(x) = (x+a_1)^{n_1}(x+a_2)^{n_2}\cdots(x+a_k)^{n_k}$

then $\displaystyle\frac{P(x)}{Q(x)} = \frac{A_{1,1}}{x+a_1} + \frac{A_{1,2}}{(x+a_1)^2} + \cdots + \frac{A_{1,n_1}}{(x+a_1)^{n_1}} + \cdots + \frac{A_{k,n_k}}{(x+a_k)^{n_k}}$

Example $\displaystyle\frac{x+1}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$

equate coefficients: 3 equations in 3 unknowns.

or: true for all $x \Rightarrow$ true for any particular x.

$x = -2$: $\quad -1 = C(-3) \quad \Rightarrow C = \frac{1}{3}$.

$x = 1$: $\quad 2 = A(9) \quad \Rightarrow A = 2/9$

$x = 0$: $\quad 1 = 4A - 2B - C \Rightarrow 2B = 4A - C - 1 = \frac{8}{9} - \frac{1}{3} - 1 = -\frac{4}{9}$

so $\displaystyle\int \frac{x+1}{(x-1)(x+2)^2}\,dx = \int \frac{2/9}{x-1} + \frac{-2/9}{x+2} + \frac{1/3}{(x+2)^2}\,dx$

$= \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| - \frac{1}{3}(x+2)^{-1} + c$