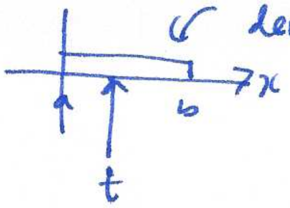
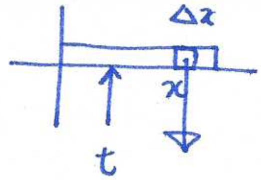
Application center of mass





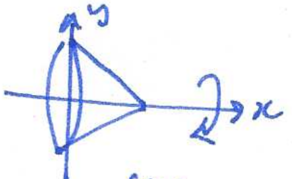
turning force $(x-t)\Delta x\,\rho(x)$

turning force at t: $\sum_i \rho(x_i)(x_i-t)\Delta x_i \approx \int_a^b x\rho(x)(x-t)\,dt$

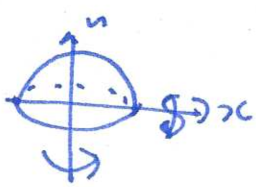
$= \int_a^b x\rho(x)\,dx - t\int_a^b \rho(x)\,dx = 0 \Rightarrow t = \dfrac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$ ← total mass.

§6.3 Volumes of revolutions

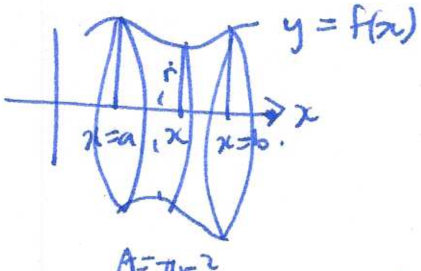
Examples



cone



half sphere



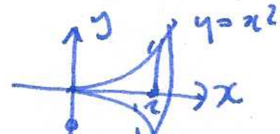
recall: $V = \int_a^b A(x)\,dx$

$A(x) = \pi r^2 = \pi y^2 = \pi (f(x))^2$

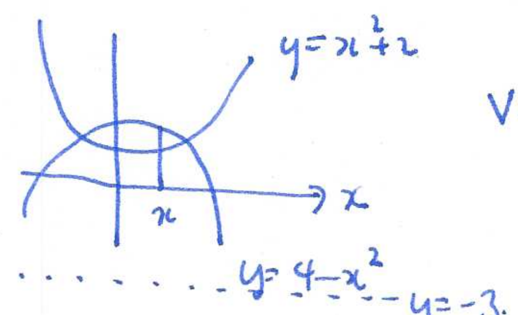
so volume $V = \int_a^b \pi (f(x))^2\,dx$

Example rotate $y = x^2$ about x axis.

$\pi \int_0^2 (x^2)^2\,dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}$



Example rotate region between $f(x) = x^2+2$, $g(x) = 4-x^2$ about $y=-3$



$V = \int_a^b A(x)\,dx$

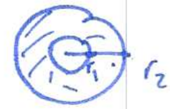
" πr^2 "

a, b: find intersection $x^2+2 = 4-x^2$

$2x^2 = 2 \quad x = \pm 1$

$A(x)$ = area of cross-section

$r_1 = 4-x^2+3 \qquad r_2 = x^2+2+3.$

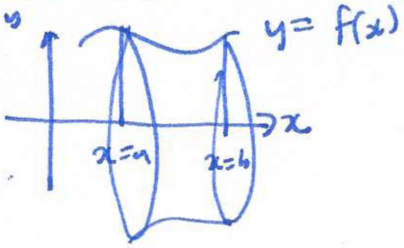


$A(x) = \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2)$

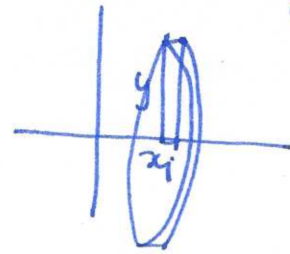
$$A(x) = \pi\left[(4-x^2+3)^2 - (x^2+5)^2\right]$$
$$(7-x^2)^2 \qquad (x^2+5)^2$$
$$49 - 14x^2 + x^4 - (x^4+10x^2+25)$$
$$24 - 4x^2$$

$$V = \pi\int_{-1}^{1} 24 - 4x^2\,dx$$
$$= \pi\left[24x - \tfrac{4}{3}x^3\right]_{-1}^{+1} = \left(48 - \tfrac{8}{3}\right)\pi.$$

Surface area



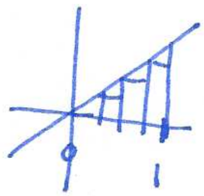
wrong : X.



area of boundary of cylinder $= 2\pi y_i \Delta x_i$.

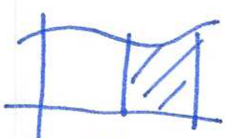
surface area $= \sum 2\pi y_i \Delta x_i$
$= \int_a^b 2\pi y\,dx$

analogy : arc length:



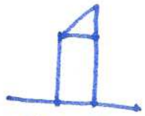
arc length $\neq \sum \Delta x_i$
$\neq \sum \Delta x_i + \Delta y_i$
$= \sum \sqrt{\Delta x_i^2 + \Delta y_i^2}$.

what goes wrong:



area of strip $\approx y(x_i)\Delta x_i$ $\qquad \frac{\text{error}}{\text{area}} \to 0$ as $\Delta x_i \to 0$

surface area is like arc length

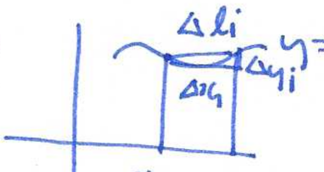


$\frac{\text{error}}{\text{arc length}} \not\to 0$ as $\Delta x_i \to 0$

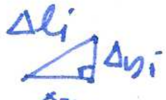
^ in fact constant $\sim \sqrt{2} - 1$

similarly $\frac{\text{error}}{\text{surface area}} \not\to 0$ as $\Delta x_i \to 0$

right way :



arc length $\approx \sum \Delta l_i$



$(\Delta l_i)^2 = (\Delta x_i)^2 + (\Delta y_i)^2$

surface area of strip $\approx 2\pi y_i \Delta l_i$

total surface area : $\sum 2\pi y_i \Delta l_i = \sum 2\pi y_i(x_i)\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ ← doesn't look like an integral!

$= 2\pi \sum y(x_i)\sqrt{1+\left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\,\Delta x_i \xrightarrow{\text{surface area}} S = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$