useful rules

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$

$f(x)$	$\int f(x)\,dx$	
x^n	$\frac{x^{n+1}}{n+1}$	$(n \neq -1)$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert$	
$\sin(x)$	$-\cos(x)$	
$\cos(x)$	$\sin(x)$	
e^x	e^x	

general rules

$f(x)$	$f'(x)$
$kf(x)$	$kf'(x)$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)\,g(x)$	$f'(x)\,g(x)+f(x)\,g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{gf'+fg'}{g^2}$
$f(g(x))$	$f'(g(x)) \cdot g'(x)$

$f(x)$	$\int f(x)\,dx$
$kf(x)$	$k\int f(x)\,dx$
$f(x)+g(x)$	$\int f(x)\,dx + \int g(x)\,dx$

§5.7 Integration by substitution

$$\int_{x=a}^{x=b} f(x)\,dx \,, \quad x(u) \quad \rightsquigarrow \quad \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\,\frac{dx}{du}\,du$$

recall : $\frac{dx}{du} = \frac{1}{\frac{du}{dx}}$ integration by substition is "reverse chain rule"

Examples • $\int x \sin(x^2)\,dx$ try: $x^2 = u$ $\frac{du}{dx} = 2x$

$$\int x\sin(x^2)\,dx = \int x \sin(u)\frac{1}{2x}\,du = \int \frac{1}{2}\sin(u)\,du = -\frac{1}{2}\cos(u)+c = -\frac{1}{2}\cos(x^2)+c$$

check!

- $\int e^{4x}\,dx$ $\quad u=4x \quad \frac{du}{dx}=4 \quad \int e^u \frac{1}{4}\,du = \frac{1}{4}e^u + c = \frac{1}{4}e^{x} + c$ $\quad$ check!

- $\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$ $\quad$ [hint: $\frac{d}{dx}\left(\ln(f(x))\right) = \frac{1}{f(x)}\cdot f'(x)$]

try $u=\cos\theta \quad \frac{du}{d\theta} = -\sin\theta \quad \int \frac{\sin\theta}{u}\cdot \frac{-1}{\sin\theta}\,du = -\int \frac{1}{u}\,du = -\ln|u|+c$

$= -\ln|\cos x| + c = \ln|\sec x| + c$

- $\int x\sqrt{x+1}\,dx$ $\quad$ try $u=x+1 \quad \frac{du}{dx}=1 \quad \int x\sqrt{u}\,du = \int (u-1)u^{1/2}\,du$

$= \int u^{3/2} - u^{1/2}\,du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$

<u>Definite integrals</u>

$\int_0^2 x^2\sqrt{x^3+1}\,dx \quad u=x^3+1 \quad \frac{du}{dx}=3x^2 \quad \int_1^9 x^2\sqrt{u}\,\frac{1}{3x^2}\,du = \int_1^9 \frac{1}{3}u^{1/2}\,du$

$= \left[\frac{2}{9}u^{3/2}\right]_1^9 = \frac{2}{9}\cdot 27 - \frac{2}{9}$

§5.8 More functions

<u>inverse trig functions</u>

$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \quad \rightsquigarrow \quad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)+c$

$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x)+c$

<u>Example</u> $\int_0^1 \frac{1}{4+x^2}\,dx$ $\quad$ try $x=2u \quad x^2=4u^2 \quad \frac{dx}{du}=2 \quad \rightsquigarrow \int_0^{1/2} \frac{1}{4+4u^2}\cdot 2\,du = \frac{1}{2}\int_0^{1/2} \frac{1}{1+u^2}\,du$

$= \left[\frac{1}{2}\tan^{-1}(u)\right]_0^{1/2} = \frac{1}{2}\tan^{-1}(1/2).$

exponential functions

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x \, dx = e^x + c$$

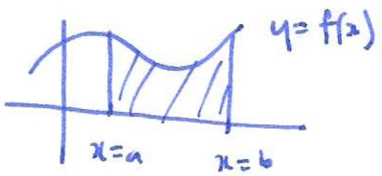
$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x\ln(b)}\right) = \ln(b)\, e^{x\ln(b)} = \ln(b)\, b^x$$

$$\int b^x \, dx = \frac{1}{\ln(b)} b^x + c$$

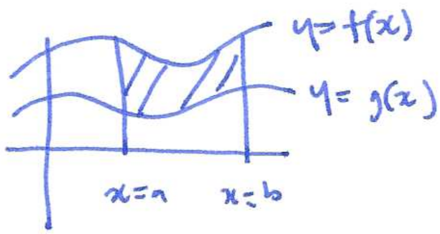
Examples $\displaystyle\int_0^1 10^x \, dx \qquad \int_0^{\pi/2} \cos\theta \cdot 10^{\sin\theta} \, d\theta$

§6.1 Area between two curves

recall



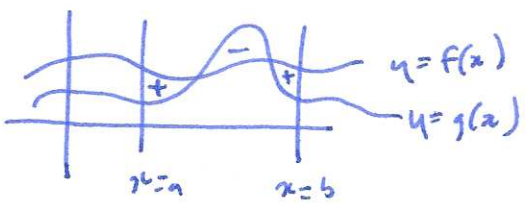
area under graph $\displaystyle\int_a^b f(x)\,dx$



area between two curves

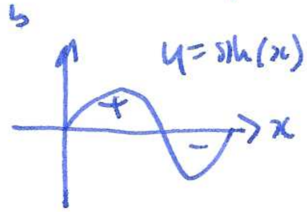
$$\int_a^b f(x) - g(x)\, dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

warning: signs!



$$\int_a^b f(x) - g(x)\, dx$$

Q: what do you do if you want "absolute/geometric area" / unsigned area?



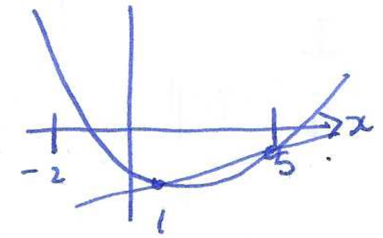
$$\int_0^{2\pi} \sin(x)\,dx = 0 \qquad \text{want: } \int_0^{2\pi} |\sin(x)|\,dx$$ ← need to know when $\sin(x)$ +ve/−ve.

i.e. need to find zeros of $\sin(x) = 0$ $\qquad \displaystyle\int_0^{\pi} \sin(x)\,dx + \int_{\pi}^{2\pi} -\sin(x)\,dx$

$$= \Big[-\cos(x)\Big]_0^{\pi} - \Big[\cos(x)\Big]_0^{2\pi} = 2 - (-2) = 4$$

Examples · find area between graphs $f(x) = x^2 - 5x - 7$ on $[-2,5]$ and $g(x) = x - 12$

draw picture:



intersection points: $x^2 - 5x - 7 = x - 12$

$x^2 - 6x + 5 = 0$

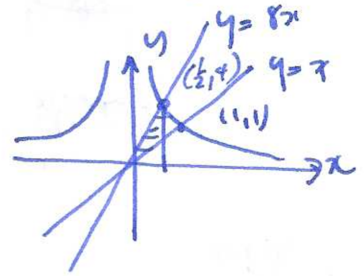
$(x-5)(x-1) = 0$

$x = 1, 5$

need to break into two integrals. $\int_{-2}^{1} f - g \, dx + \int_{1}^{5} g - f \, dx$

· find area bounded by $y = \frac{1}{x^2}$, $y = x$, $y = 8x$

draw picture:

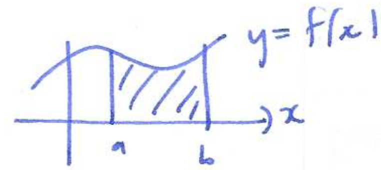


← need to break into 2 integrals

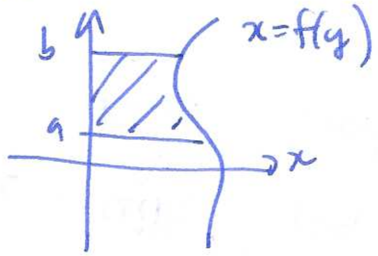
$$\int_0^{1/2} 8x - x \, dx + \int_{1/2}^{1} \frac{1}{x^2} - x \, dx$$

Integration along y-axis

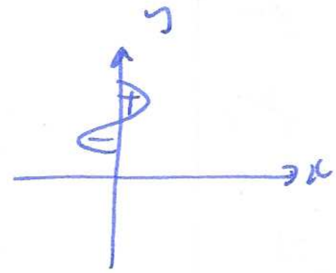
recall: $\int_a^b f(x) \, dx$



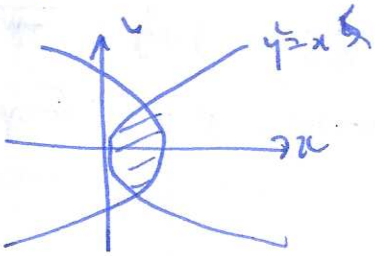
$\int_a^b f(y) \, dy$



signed area:



Example find area between the curves $x = y^2$ and $x = 1 - y^2$



intersection points: $y^2 = 1 - y^2$ $2y^2 = 1$ $y = \pm \frac{1}{\sqrt{2}}$

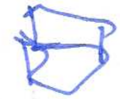
$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} 1 - y^2 - y^2 \, dy$$

§6.2 Volume, average value

A = area of base

recall: volume of cylinder is Ah

Fact: this works for cylinders of any base shape:



$V = Ah$.

suppose shape is not a cylinder.

can approximate by horizontal slices.