Math 232 Calculus 2 Spring 25 Midterm 2b Part 1 of 2

Name: Solutions

- I will count your best 8 of the following 10 questions from Parts 1 (Q1–5) and Part 2 (Q6–10).
- You may use a calculator, and a US letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (10 points) Find $\int \sin 5x \sin 3x \; dx$.

$\cos(A+B) = \cos(A)\cos B - \sin A \sin B$

$\cos(A-B) = \cos(A)\cos B + \sin A \sin B$

$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B$

$\frac{1}{2}\int \cos(2x) - \cos(8x)\,dx = \frac{1}{4}\sin 2x - \frac{1}{16}\sin(8x) + c$

$u = \sin x$

$\frac{du}{dx} = \cos x$

(2) (10 points) Find $\int \cos x \sin^2 x \; dx$.

$\int \cos x \; u^2 \frac{dx}{du} du = \int \cos x \; u^2 \frac{1}{\cos x} du = \int u^2 du = \frac{1}{3} u^3 + C$

$= \frac{1}{3} \sin^3 x + C$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

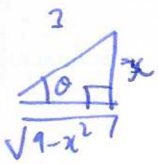
(3) (10 points) Find $\displaystyle\int \frac{1}{x\sqrt{9-x^2}}\,dx$.

$x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

$$\int \frac{1}{3\sin\theta\sqrt{9-9\sin^2\theta}} \cdot \frac{dx}{d\theta}\,d\theta = \int \frac{1}{3\sin\theta\, 3\cos\theta} \cdot 3\cos\theta\,d\theta = \int \frac{1}{3}\operatorname{cosec}\theta\,d\theta$$

$$= \frac{1}{3}\ln\left|\cot\theta - \operatorname{cosec}\theta\right| + c$$



$$= \frac{1}{3}\ln\left|\frac{\sqrt{9-x^2}}{x} - \frac{3}{x}\right| + c$$

(4) (10 points) Find $\displaystyle\int \frac{3}{(x^2+2)(x-1)}\,dx$.

$$\frac{3}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1} = \frac{(Ax+B)(x-1)+C(x^2+2)}{(x^2+2)(x-1)}$$

$x=1: \quad 3 = 3C \qquad C = 1$

$x=0: \quad 3 = -B+2C \qquad B=-1 \qquad \cancel{B = \frac{6}{11} - 3 = -\frac{27}{11}}$

$x=-1: \quad 3 = (-A-1)(-2)+3 \qquad A=-1$

$$\int \frac{-x-1}{x^2+2} + \frac{1}{x-1}\,dx = \int \frac{-x}{x^2+2} - \frac{1}{x^2+2}\,dx + \ln|x-1|$$

$$= -\frac{1}{2}\ln|x^2+2| + \ln|x-1| - \int \frac{1}{x^2+2}\,dx \qquad x=\sqrt{2}u, \quad \frac{dx}{du} = \sqrt{2}$$

$$\int \frac{1}{2(u^2+1)}\frac{dx}{du}\,du = \frac{\sqrt{2}}{2}\int \frac{1}{1+u^2}\,du = \frac{\sqrt{2}}{2}\tan^{-1}(u)+C$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

$$= -\frac{1}{2}\ln|x^2+2| + \ln|x-1| - \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

$$\int u'v\,dx = uv - \int uv'\,dx$$

(5) (10 points) Find $\displaystyle\int_{-\infty}^{0} e^{3x}\sin(x)\,dx$.

$$\int \underbrace{e^{3x}}_{u'}\underbrace{\sin x}_{v}\,dx = \frac{1}{3}e^{3x}\sin x - \int \underbrace{\frac{1}{3}e^{3x}}_{u'}\underbrace{\cos x}_{v}\,dx$$

$$= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \int \frac{1}{9}e^{3x}\sin x\,dx.$$

$$\frac{10}{9}\int e^{3x}\sin x\,dx = \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x + C$$

$$\int_{-\infty}^{0} e^{3x}\sin x\,dx = \lim_{R\to-\infty}\frac{9}{10}\left[\frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x\right]_R^0.$$

$$= -\frac{9}{10}\,\frac{1}{9} = -\frac{1}{10}$$

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Midterm 2	
Overall	

(6) (10 points) Find the degree three Taylor polynomial for $f(x) = \ln(1 + x^2)$ centered at $x = 0$, by any method.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\ln|1-x| = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$\ln|1+x^2| = x^2 - \frac{x^4}{2} + \cdots$$

$$T_3(x) = x^2$$

(7) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2+5n+6}$ converge or diverge? If it converges, find the exact value.

$n=-2: \quad 1 = A$
$n=-3: \quad 1 = -B.$

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3} = \frac{A(n+3)+B(n+2)}{(n+2)(n+3)}$$

$$\sum_{n=2}^{\infty} \frac{1}{n+2} - \frac{1}{n+3} \Rightarrow S_n = \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{n+2} - \frac{1}{n+3} = \frac{1}{4} - \frac{1}{n+3}$$

$\lim_{n\to\infty} S_n = \frac{1}{4}$ converges to $\frac{1}{4}$

(8) Does the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$ converge or diverge?

~~comparison tes~~ limit comparison test with $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left| \frac{n}{\sqrt{n^3+2}} \cdot \sqrt{n} \right| = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2/n^3}} = 1 \quad 0 < \quad < \infty$$

So $\sum a_n$ converges iff $\sum b_n$ converges.

$\sum \frac{1}{\sqrt{n}}$ diverges (p-series $p<\frac{1}{2}$) so $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$ diverges

(9) Does the series $\sum_{n=2}^{\infty} \frac{n}{n^2-2}$ converge or diverge?

comparison test $\quad n^2-2 < n^2$

$$\frac{1}{n^2-2} > \frac{1}{n^2}$$

$$\frac{n}{n^2-2} > \frac{n}{n^2} > \frac{1}{n}$$

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series, ~~p-test~~ series $p \le 1$).

so $\sum_{n=2}^{\infty} \frac{n}{n^2-2}$ diverges

(10) (a) Write down the power series for $\cos(x)$ centered at $x = 0$.
(b) Use this to write down a power series for $\cos(\sqrt{x})$ centered at $x = 0$.
(c) Find the radius of convergence of this power series.
(d) Explain why this method doesn't work for $\sin(\sqrt{x})$.

a) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

b) $\cos(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

c) ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(2n+2)!} \frac{(2n)!}{x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0 < 1$

so converges for all x, radius of convergence $R = \infty$.

d) $\sin(x) = x - \frac{x^3}{3!} + \cdots$

so $\sin(\sqrt{x}) = \sqrt{x} - \frac{x^{3/2}}{3!} - \cdots$ ← not power series in x.