Math 232 Calculus 2 spring 25 Midterm 2a Part 1 of 2

Name: Solutions

- I will count your best 8 of the following 10 questions from Parts 1 (Q1–5) and 2 (Q6–10).
- You may use a calculator, and a US letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (10 points) Find $\int \cos 2x \cos 5x \, dx$.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\frac{1}{2} \int \cos(7x) + \cos(3x) \, dx$

$\frac{1}{14} \sin(7x) + \frac{1}{6} \sin(3x) + c$

(2) (10 points) Find $\int \sin x \cos^2 x \, dx$.

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$$\int \sin x \, u^2 \frac{dx}{du} du = \int \sin x \, u^2 \frac{1}{-\sin x} du$$

$$= \int -u^2 du = -\frac{1}{3}u^3 + c = -\frac{1}{3}\cos^3 x + c$$

(3) (10 points) Find $\int \frac{1}{x\sqrt{4-x^2}}\,dx$.

$x = 2\sin\theta$

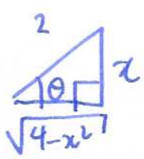
$\frac{dx}{d\theta} = 2\cos\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$\int \frac{1}{2\sin\theta\sqrt{4-4\sin^2\theta}}\frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{1}{2\sin\theta\, 2\cos\theta}\, 2\cos\theta\, d\theta = \frac{1}{2}\int \operatorname{cosec}\theta\, d\theta = \frac{1}{2}\ln\left|\cot\theta - \operatorname{cosec}\theta\right| + c$$



$$= \frac{1}{2}\ln\left|\frac{\sqrt{4-x^2}}{x} - \frac{2}{x}\right| + c$$

(4) (10 points) Find $\int \frac{3x+1}{(x-2)(x^2+3)}\,dx$.

$$\frac{3x+1}{(x-2)(x^2+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3)+(Bx+C)(x-2)}{(x-2)(x^2+3)}.$$

$x=2$: $7 = 7A$, $\Rightarrow A = 1$

$x=0$: $1 = 3A - 2C = 3 - 2C \Rightarrow C = 1$

$x=1$: $4 = 4A + (B+C)(-1)$

$4 = 4 + (B+1)(-1)$. $B = -1$

$$\int \frac{1}{x-2} + \frac{-x+1}{x^2+3}\,dx = \ln|x-2| + \int \frac{-x}{x^2+3} + \frac{1}{x^2+3}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+3| + \underbrace{\int \frac{1}{x^2+3}\,dx}$$

$x = \sqrt{3}u$

$\frac{dx}{du} = \sqrt{3}$.

$$\int \frac{1}{3u^2+3}\frac{dx}{du}\,du = \frac{\sqrt{3}}{3}\int \frac{1}{1+u^2}\,du = \frac{\sqrt{3}}{3}\tan^{-1}(u)$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+3| + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$\int u'v\,dx = uv - \int uv'\,dx$$

(5) (10 points) Find $\int_{-\infty}^{0} e^{2x}\cos(x)\,dx$.

$$\int \underbrace{e^{2x}}_{u'}\underbrace{\cos(x)}_{v}\,dx = \frac{1}{2}e^{2x}\cos(x) - \int \frac{1}{2}e^{2x}\cdot(-\sin x)\,dx$$

$$= \frac{1}{2}e^{2x}\cos(x) + \int \frac{1}{2}\underbrace{e^{2x}}_{u'}\underbrace{\sin x}_{v}\,dx$$

$$\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin x - \int \frac{1}{4}e^{2x}\cos x\,dx$$

$$\frac{5}{4}\int e^{2x}\cos(x)\,dx = \frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin x$$

$$\int_{-\infty}^{0} e^{2x}\cos(x)\,dx = \frac{4}{5}\lim_{R\to-\infty}\left[\frac{1}{2}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin x\right]_R^0$$

$$= \frac{4}{5}\cdot\frac{1}{2} = \frac{2}{5}$$

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Midterm 2	
Overall	

(6) (10 points) Find the degree three Taylor polynomial for $f(x) = \ln(1+x^2)$ centered at $x = 0$, by any method.

$f(x) = \ln(1+x^2)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} \cdot 2x$ $\qquad f'(0) = 0$

$f''(x) = \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$ $\qquad f''(0) = 2$

$f^{(3)}(x) = \frac{(1+x^2)^2(-4x) - 2(1+x^2)(2x)(2-2x^2)}{(1+x^2)^4}$ $\qquad f^{(3)}(0) = 0$

$T_3(x) = \frac{2x^2}{2!} = x^2$

(7) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2+3n+2}$ converge or diverge? If it converges, find the exact value.

$$\frac{1}{n^2+3n+2} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$$

$n=-1: \quad 1 = +A$

$n=-2: \quad 1 = -B.$

$$\sum_{n=2}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow S_n = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n+1} - \frac{1}{n+2}$$

$$S_n = \frac{1}{3} - \frac{1}{n+2}$$

$\lim_{n\to\infty} S_n = \frac{1}{3}$ converges to $\frac{1}{3}$

(8) Does the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+3}}$ converge or diverge?

limit comparison test with $b_n = \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \frac{n\sqrt{n}}{\sqrt{n^3+3}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+3}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+3/n^3}} = 1 < \infty$$

$0 <$

so $\sum a_n$ converges iff $\sum b_n$ converges

$\sum \frac{1}{\sqrt{n}}$ diverges (p-series $p < \frac{1}{2}$)

so $\sum \frac{n}{\sqrt{n^3+3}}$ diverges.

(9) Does the series $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-3}$ converge or diverge?

limit comparison test with $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left| \frac{\sqrt{n}}{n^2-3} \cdot n^{3/2} \right| = \lim_{n\to\infty} \frac{n^2}{n^2-3} = \lim_{n\to\infty} \frac{1}{1-3/n^2} = 1 \quad {}_{0<} 1 < \infty$$

so $\sum a_n$ converges iff $\sum b_n$ converges

$\sum \frac{1}{n^{3/2}}$ converges (p-test $p > 1$).

So $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-3}$ converges.

(10) (a) Write down the power series for $\cos(x)$ centered at $x = 0$.
(b) Use this to write down a power series for $\cos(\sqrt{x})$ centered at $x = 0$.
(c) Find the radius of convergence of this power series.
(d) Explain why this method doesn't work for $\sin(\sqrt{x})$.

a) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

b) $\cos(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

c) ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(2n+2)!} \frac{(2n)!}{x^n} \right| = |x| \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0 < 1$

so converges for all x, radius of convergence $R = \infty$.

d) $\sin(x) = x - \frac{x^3}{3!} + \cdots$

so $\sin(\sqrt{x}) = \sqrt{x} - \frac{x^{3/2}}{3!} - \cdots$ ← not power series in x.