# MT2 Sample Solutions



**Q1** $\int \cos^3 x \, dx = \int \cos x \,(1-\sin^2 x)\, dx$ $\quad u=\sin x$, $\frac{du}{dx}=\cos x$. $\quad \int \cos x\,(1-u^2)\,\frac{dx}{du}\,du$

$= \int \cos x\,(1-u^2)\,\frac{1}{\cos x}\,du = \int 1-u^2\,du = u-\frac{1}{3}u^3+c = \sin x - \frac{1}{3}\sin^3 x + c.$

**Q2** $\int \cos 6x \sin 5x \, dx$

$\left.\begin{array}{l}\sin(A+B) = \sin A\cos B + \cos A \sin B\\ \sin(A-B) = \sin A\cos B - \cos A\sin B\end{array}\right\}$ $\sin(A+B)+\sin(A-B) = 2\sin A\cos B.$

$=\frac{1}{2}\int \sin(-x)+\sin(11x)\,dx = \frac{1}{2}\cos x - \frac{1}{22}\cos 11x + c$

**Q3** $\int \frac{x}{\sqrt{x^2+16}}\,dx$

$\sin^2 x+\cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$

set $4x = \tan\theta$
$4\frac{dx}{d\theta} = \sec^2\theta.$

$\int \frac{\frac{1}{4}\tan\theta}{\sqrt{16\tan^2\theta+16}}\frac{dx}{d\theta}\,d\theta = \frac{1}{16}\int \frac{\tan\theta}{\sec\theta}\sec^2\theta\,d\theta = \int \tan\theta\sec\theta\,d\theta = \sec\theta + c$

$= \sqrt{\tan^2\theta+1}+c = \sqrt{1+16x^2}+c$

**Q4** $\int \frac{4-x^2}{(x+2)^2(x-1)}\,dx \quad \int \frac{(2-x)(2+x)}{(x+2)^2(x-1)}\,dx = \int \frac{2-x}{(x+2)(x-1)}\,dx$

$\frac{-x+2}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}$ $\quad -x+2 = A(x-1)+B(x+2)$

$x=1: \quad 1 = 3B \quad B = 1/3$
$x=-2: \quad -4 = -3A \quad A = 4/3$

$\int \frac{4/3}{x+2}+\frac{1/3}{x-1}\,dx = \frac{4}{3}\ln|x+2| + \frac{1}{3}\ln|x-1|+c$

**Q5** $\int_0^1 x^3\ln(x^4)\,dx$ $\quad u=x^4$, $\frac{du}{dx}=4x^3$ $\quad \int_0^1 x^3\ln(u)\frac{dx}{du}\,du = \int_0^1 x^3\ln(u)\frac{1}{4x^3}\,du$

$=\frac{1}{4}\int_0^1 \ln(u)\,du = \lim_{R\to 0}\frac{1}{4}\int_R^1 \ln(u)\,du = \lim_{R\to 0}\frac{1}{4}\left[\frac{1}{u}\right]_R^1$ DNE.

**Q6** $\int_0^2 \frac{x}{1-x}\,dx$ $\quad -x+1\,\overline{)\,x\quad}^{\,-1}$ $\quad \frac{x-1}{+1}$ $\qquad \int_0^2 -1+\frac{1}{1-x}\,dx = -x-\ln|1-x|+c$

$= \underbrace{\lim_{R\to 1^-}\int_0^R \frac{x}{1-x}\,dx}\ + \lim_{R\to 1^+}\int_R^2 \frac{x}{1-x}\,dx.$

$= \lim_{R\to 1}\Big[-x-\ln|1-x|\Big]_0^R$ DNE.

**Q7** $\int_0^\infty \frac{1}{4x^2+1}\,dx$ $\qquad 2x=u$, $\frac{dx}{du}=\frac{1}{2}$ $\qquad \int \frac{1}{u^2+1}\frac{dx}{du}\,du = \frac{1}{2}\int\frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1}(u)+c$

$= \int_0^\infty \frac{1}{2}\frac{1}{u^2+1}\,du = \lim_{R\to\infty}\frac{1}{2}\int_0^R \frac{1}{u^2+1}\,du = \frac{1}{2}\lim_{R\to\infty}\Big[\tan^{-1}(u)\Big]_0^R = \frac{\pi}{4}$

**Q8** No: $f'(0)$ DNE.

$f(x)=x^{1/3}$ $\qquad f(1)=1$

$f'(x)=\frac{1}{3}x^{-2/3}$ $\qquad f'(1)=1/3$

$f''(x)=-\frac{2}{9}x^{-5/3}$ $\qquad f''(1)=-2/9$

$f^{(3)}(x)=\frac{10}{27}x^{-8/3}$ $\qquad f^{(3)}(1)=\frac{10}{27}$

$T_3(x) = 1+\frac{1}{3}(x-1)+-\frac{2}{9}\frac{(x-1)^2}{2!}+\frac{10}{27}\frac{(x-1)^3}{3!}$

error $\leq \frac{K}{4!}$ where $K$ is an upper bound for $|f^{(4)}(x)|$ on $[1,2]$.

$f^{(4)}(x) = -\frac{80}{81}x^{-11/3}$ ← largest value when $x=1$, so can choose $K=\frac{80}{81}<1$.

**Q9** $a_n = \frac{3^n}{n!} = \frac{3.3.3\ldots3\ 3}{\underbrace{1.2.}_{\frac{9}{2}}\underbrace{3\ldots(n-1)}_{\leq 1}\underbrace{n}} \leq \frac{9}{2}\cdot\frac{3}{n}\to 0$ as $n\to\infty$.

**Q10** $\sum_{n=1}^\infty \frac{3^n}{n!}$ use ratio test $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \lim_{n\to\infty}\frac{3}{n+1}=0<1$

so converges.

**Q11** geometric series, converges to $\frac{e^{-3}}{1-e^{-3}}$

**Q12** $\frac{1}{(n+1)(n+2)} = \frac{A}{n+1}+\frac{B}{n+2}$ $\qquad 1 = A(n+2)+B(n+1)$ $\qquad n=-1$: $1 = \cancel{5A}\ A$ $\qquad n=-2$: $1=-B$

$$\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow S_n = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{2} - \frac{1}{n+2}$$

$\lim_{n\to\infty} S_n = \frac{1}{2}$ so converges to $\frac{1}{2}$.

**Q13** $\lim_{n\to\infty} \cos\left(\frac{1}{n^3}\right) = 1 \neq 0$ so does not converge.

**Q14** $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^3} = a_n$ ~~claim for large $n$, $\frac{(\ln n)^2}{n}$~~ → limit comparison test

with $b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{\ln(n)^2}{n^3} \cdot n^2 = \lim_{n\to\infty} \frac{\ln(n)^2}{n}$

$$\lim_{x\to\infty} \frac{\ln(x)^2}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2\ln(x)\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x} = \lim_{x\to\infty} \frac{2\frac{1}{x}}{1} = 0$$

so $\sum b_n$ converges $\Rightarrow \sum a_n$ converges, so $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^3}$ converges.

**Q15** $\sum_{n=1}^{\infty} \frac{n\sin(n)}{n^4+1} = a_n$ limit comparison test with $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{n\sin(n)}{n^4+1} \cdot n^2 = \lim_{n\to\infty} \frac{\sin(n)}{n + 1/n} = 0$ so $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{n\sin(n)}{n^4+1}$ converges.

**Q16** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} = a_n$ limit comparison test with $b_n = \frac{1}{\sqrt{n}}$ (diverges).

$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ so $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges

so $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ diverges.

**Q17** $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^3} \frac{n^3}{x^n}\right| = \lim_{n\to\infty} |x| \frac{n^3}{(n+1)^3} = |x|$

so converges for $|x| < 1$.

check endpoints: $x = 1$ $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges by alternating series test.

**Q18** $f(x) = \sin(x^{1/2})$ $\qquad f(1) = \sin(1)$

$f'(x) = \cos(x^{1/2})\,\frac{1}{2}x^{-1/2}$ $\qquad f'(1) = \frac{1}{2}\cos(1)$

$f''(x) = -\sin(x^{1/2})\cdot\frac{1}{4}\frac{1}{x} + \cos(x^{1/2})\cdot -\frac{1}{4}x^{-3/2}$ $\qquad f''(1) = -\frac{\sin(1)}{4} - \frac{\cos(1)}{4}.$

$$T(x) = \sin(1) + \frac{1}{2}\cos(1)(x-1) - \frac{1}{4}(\sin(1)+\cos(1))\frac{(x-1)^2}{2!}$$

**Q19** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ $\qquad x^3e^{-x^3} = x^3\left(1 - x^3 + \frac{x^6}{2!} \ldots\right) = x^3 - x^6 + \frac{x^9}{2} \ldots$

**Q20** $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots$ $\qquad \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} \ldots$

$\frac{\sin(x^2)}{x} = x - \frac{x^5}{3!} + \frac{x^9}{5!} \ldots$ $\qquad \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(2n-1)!}(-1)^{n+1}$

**Q21**

$f(x) = x^{1/2}$ $\qquad$ ~~$t(x)$~~ $f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(4) = \frac{1}{4}$ $\qquad T(x) = 2 + \frac{1}{4}(x-4) + -\frac{1}{32}\frac{(x-4)^2}{2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f''(4) = -\frac{1}{32}$ $\qquad T(3) = 2 - \frac{1}{4} - \frac{1}{32}\cdot\frac{1}{2}.$

error bound $\leq \frac{K|4-3|^3}{3!}$ when $K$ is an upper bound for $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$

on $[3,4]$ max value $\frac{3}{8}\cdot\frac{1}{3^{5/2}} = K.$