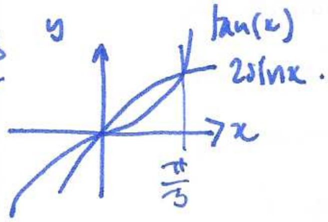
SMT1 Solutions

Q1 $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx$ sub $u=\sin x$, $\dfrac{du}{dx}=\cos x$ $\displaystyle\int \frac{\cos x}{1-u}\frac{dx}{du}\,du = \int \frac{\cos x}{1-u}\,\frac{1}{\cos x}\,du = \int \frac{1}{1-u}\,du$

$= -\ln|1-u|+c = -\ln|1-\sin x|+c$

Q2 $\displaystyle\int \frac{\cos x}{1-\sin^2 x}\,dx$ ~~sub u=~~ $1-\sin^2 x=\cos^2 x$ $\displaystyle\int \frac{\cos x}{\cos^2 x}\,dx = \int \sec x\,dx = \ln|\sec x+\tan x|+c$

Q3

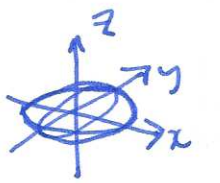


$\tan(x)=\dfrac{\sin x}{\cos x}=2\sin x \Rightarrow \cos x=\dfrac{1}{2} \Rightarrow x=\dfrac{\pi}{3}.$

$$\int_{1/2}^{\pi/3} 2\sin x-\tan x\,dx + \int_{-\pi/3}^{3/2} \tan x-2\sin x\,dx$$

$$= \Big[-2\cos(x)-\ln|\sec x|\Big]_{1/2}^{\pi/3} + \Big[\ln|\sec x|+2\cos(x)\Big]_{-\pi/3}^{3/2} \simeq 1.29$$

Q4

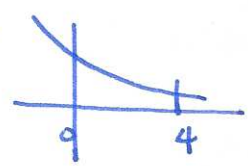


$\perp$ z-axis: cross section a circle of radius $x^2+y^2=16-9z^2$ so $r=\sqrt{16-9z^2}$.

$$V=\int A(z)\,dz = \int_{-4/3}^{4/3}\pi r^2\,dz = \int_{-4/3}^{4/3}\pi\left(16-9z^2\right)dz = \pi\Big[16z-3z^3\Big]_{-4/3}^{4/3}$$

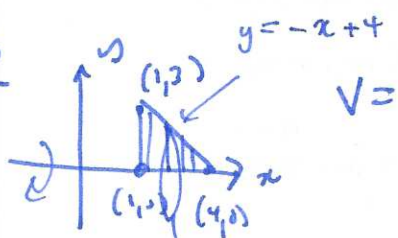
$$= \pi\left(\frac{64}{3}-3\cdot\frac{64}{27}\right) = \pi\,\frac{64}{3}\left(1-\frac{1}{3}\right) = \frac{128\pi}{3}.$$

Q5



$$\frac{1}{4}\int_0^4 e^{-4x}\,dx = \frac{1}{4}\Big[-\frac{1}{4}e^{-4x}\Big]_0^4 = \frac{1}{16}\left(1-e^{-16}\right).$$

Q6



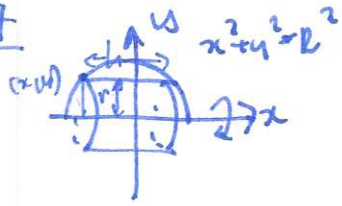
$r=y=-x+4.$

$$V=\int A(x)\,dx = \int_1^4 \pi r^2\,dx = \int_1^4 \pi(4-x)^2\,dx = \pi\int_1^4 16-8x+x^2\,dx$$

$$=\pi\Big[16x-4x^2+\tfrac{1}{3}x^3\Big]_1^4 = \pi\left(64-64+\tfrac{1}{3}64-\left(16-4+\tfrac{1}{3}\right)\right)$$

$$=\pi\left(\frac{64}{3}-\frac{37}{3}\right)=\pi\frac{27}{3}=9\pi.$$

Q7



$$V=\int A(y)\,dy = \int_0^R 2\pi r h\,dy = \int_0^R 2\pi y\,2\sqrt{R^2-y^2}\,dy$$

sub $u = R^2 - y^2$

$\frac{du}{dy} = -2y$

$4\pi \int_{R^2}^{0} y\sqrt{u}\,\frac{dy}{du}\,du = 4\pi \int_{R^2}^{0} y\sqrt{u}\,\frac{1}{-2y}\,du = 2\pi \int_{0}^{R^2} u^{1/2}\,du$

$= 2\pi \left[\frac{2}{3} u^{3/2} \right]_0^{R^2} = \frac{4}{3}\pi R^3$

Q8

$\int \underbrace{x^2}_{v'} \underbrace{\ln(x-2)}_{u}\,dx$ $u' = \frac{1}{x-2}$ $v = \frac{1}{3}x^3$ $= \frac{1}{3}x^3 \ln(x-2) - \int \frac{x^3}{3(x-2)}\,dx$

$$\begin{array}{r} x^2 + 2x + 4 \\ x-2 \,\overline{)\, x^3 \quad\quad\quad} \\ x^3 - 2x^2 \\ \hline 2x^2 \\ 2x^2 - 4x \\ \hline 4x \\ 4x - 8 \\ \hline 8 \end{array}$$

$= \frac{1}{3}x^3 \ln(x-2) - \frac{1}{3}\int x^2 + 2x + 4 + \frac{8}{x-2}\,dx$

$= \frac{1}{3}x^3 \ln(x-2) - \frac{1}{9}x^3 - \frac{x^2}{3} - \frac{4}{3}x - \frac{8}{3}\ln|x-2| + c$

Q9

$\int \underbrace{e^{-3x}}_{u} \underbrace{\cos(2x)}_{v'}\,dx = e^{-3x}\cdot\frac{1}{2}\sin 2x + \int \underbrace{\frac{3}{2}e^{-3x}}_{u} \underbrace{\sin 2x}_{v'}\,dx$ $u' = -3e^{-3x}$ $v = -\frac{1}{2}\cos 2x$

$u' = -3e^{-3x}$ $v = \frac{1}{2}\sin 2x$

$\int e^{-3x}\cos(2x)\,dx = \frac{1}{2}e^{-3x}\sin 2x + -\frac{3}{4}e^{-3x}\cos 2x - \int \frac{9}{4}e^{-3x}\cos 2x\,dx$

$\int e^{-3x}\cos(2x)\,dx = \frac{4}{13}\left(\frac{1}{2}e^{-3x}\sin 2x - \frac{3}{4}e^{-3x}\cos 2x \right) + c$

Q10 1 $\int \underbrace{x}_{u} \underbrace{e^{-x}\sin(x)}_{v'}\,dx = x\int e^{-x}\sin(x)\,dx - \int \int e^{-x}\sin x\,dx\,dx + c$

$u' = 1$ $v = \int e^{-x}\sin(x)\,dx$

$\int \underbrace{e^{-x}}_{u} \underbrace{\sin x}_{v'}\,dx = e^{-x}\cdot -\cos x - \int \underbrace{-e^{-x}}_{u}\cdot \underbrace{-\cos x}_{v'}\,dx$

$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx$

$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}\cos x - \frac{1}{2}e^{-x}\sin x.$

1 $= -\frac{1}{2}xe^{-x}(\sin x + \cos x) + \frac{1}{2}\int e^{-x}(\cos x + \sin x)\,dx.$ $= \frac{1}{2}\int e^{-x}\cos x\,dx + \frac{1}{2}\int e^{-x}\sin x\,dx$

already done this.

$$\int \underbrace{e^{-x}}_{u}\underbrace{\cos x}_{v'}\,dx = e^{-x}\sin x + \int \underbrace{e^{-x}}_{u}\underbrace{\sin x}_{v'}\,dx = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

$$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x)$$

So ⊛ $= -\frac{1}{2}xe^{-x}(\sin x + \cos x) \cancel{+} - \frac{1}{4}e^{-x}(\cos x - \sin x) + \frac{1}{4}e^{-x}(\sin x - \cos x) + c.$

Q11 $\displaystyle\int_0^{\pi/2} \sin^2(x)\cos^5(x)\,dx$ sub $u = \sin x$, $\dfrac{du}{dx} = \cos x$ $\displaystyle\int_0^{\pi/2} u^2 \cos^4(x)\cos(x)\frac{dx}{du}\,du.$

$$= \int_0^{\pi/2} u^2(1-\sin^2 x)^2\cos(x)\frac{1}{\cos x}\,du = \int u^2(1-u^2)^2\,du = \int u^2(1-2u^2+u^4)\,du$$

$$= \int u^2 - 2u^4 + u^6\,du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + c = \left[\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x \cancel{+c}\right]_0^{\pi/2}.$$

$$= \frac{1}{3} - \frac{2}{5} + \frac{1}{7}.$$

Q12 $\displaystyle\int \sin(4x)\cos(5x)\,dx$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B.$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B.$

$$= \frac{1}{2}\int \sin(9x) + \sin(-x)\,dx = -\frac{1}{18}\cos(9x) + \cos x + c.$$

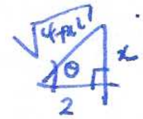
Q13 $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$ sub $x = 2\tan\theta$, $\dfrac{dx}{d\theta} = 2\sec^2\theta$ $\displaystyle\int \frac{2\tan^2\theta}{\sqrt{4\tan^2\theta + 4}}\frac{dx}{d\theta}\,d\theta$

$$= \int \frac{2\tan^2\theta}{2\sqrt{\tan^2\theta+1}}2\sec^2\theta\,d\theta = 2\int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta = 2\int \tan^2\theta\sec\theta\,d\theta \quad ⊛$$

$$\int \underbrace{\tan\theta}_{u}\underbrace{\tan\theta\sec\theta}_{v'}\,d\theta = \tan\theta\sec\theta - \int \sec^2\theta\sec\theta\,d\theta = \tan\theta\sec\theta - \int (1+\tan^2\theta)\sec\theta\,d\theta$$

$$2\int \tan^2\theta\sec\theta\,d\theta = \tan\theta\sec\theta - \int \sec\theta\,d\theta = \tan\theta\sec\theta - \ln|\sec\theta + \tan\theta| + c$$

⊛ $= \tan\theta\sec\theta - \ln|\sec\theta + \tan\theta| + c.$ $\tan\theta = \dfrac{x}{2}$



$$= \frac{x}{2}\cdot\frac{\sqrt{4+x^2}}{2} - \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c.$$

Q14 $\int \sqrt{9x^2-1}\,dx$ $\quad x = \frac{1}{3}\operatorname{cosec}\theta \quad \frac{dx}{d\theta} = -\frac{1}{3}\operatorname{cosec}\theta\cot\theta \quad \int \sqrt{\operatorname{cosec}^2\theta - 1}\cdot -\frac{1}{3}\operatorname{cosec}\theta\cot\theta\, d\theta$

$= \int \sqrt{\cot^2\theta}\cdot -\frac{1}{3}\operatorname{cosec}\theta\cot\theta\,d\theta = -\frac{1}{3}\int \operatorname{cosec}\theta\cot^2\theta\,d\theta$. ⊛ .

$\frac{1}{3x} = \sin\theta$

$\int \underbrace{\operatorname{cosec}\theta\cot\theta}_{v'}\ \underbrace{\cot\theta}_{u}\,d\theta = -\operatorname{cosec}\theta\cot\theta - -\int \operatorname{cosec}\theta\,\underset{1+\cot^2\theta}{\underset{\parallel}{\operatorname{cosec}^2\theta}}\,d\theta$

$2\int \operatorname{cosec}\theta\cot^2\theta\,d\theta = -\operatorname{cosec}\theta\cot\theta - \int\operatorname{cosec}\theta\,d\theta$

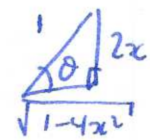
$= -\operatorname{cosec}\theta\cot\theta + \ln\left|\operatorname{cosec}\theta + \cot\theta\right| + c$.

⊛ $= +\frac{1}{3}\operatorname{cosec}\theta\cot\theta - \frac{1}{3}\ln\left|\operatorname{cosec}\theta+\cot\theta\right| + c$

$= \frac{1}{3}\,3x\sqrt{9x^2-1} - \frac{1}{3}\ln\left|3x + \sqrt{9x^2-1}\right| + c$.

Q15 $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ $\quad x = \frac{1}{2}\sin\theta \quad \frac{dx}{d\theta} = \frac{1}{2}\cos\theta \quad \int \frac{\frac{1}{2}\sin\theta}{\sqrt{1-\underset{\sin}{\cancel{\cos^2}}\theta}}\frac{dx}{d\theta}\,d\theta = \int \frac{\frac{1}{2}\sin\theta}{\sqrt{\cos^2\theta}}\,\frac{1}{2}\cos\theta\,d\theta$

$= \frac{1}{4}\int \sin\theta\,d\theta = -\frac{1}{4}\cos\theta + c = -\frac{1}{4}\sqrt{1-4x^2} + c$.



Q16 $\int \tan^3 2x\,dx$ $\quad u = 2x \quad \frac{du}{dx} = 2 \quad \int \tan^3 u\,\frac{dx}{du}\,du = \frac{1}{2}\int \tan^3 u\,du$.

$= \frac{1}{2}\int \tan u\,.\,\tan^2 u\,du = \frac{1}{2}\int \tan u\left(\sec^2 u - 1\right)du = \int \tan u\sec^2 u - \underbrace{\tan u\,du}_{-\ln|\sec u| + c}$.

see Q13 .

Q17 $\int \frac{5x-1}{x^2-x-2}\,dx$ $\quad \frac{5x-1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{(x+1)(x-2)}$

$x = 2: \quad 9 = 3A \quad A = 3$

$x = -1 \quad -6 = -3B \quad B = 2$

$\int \frac{3}{x-2} + \frac{2}{x+1}\,dx = 3\ln|x-2| + 2\ln|x+1| + c$.

Q18

$$
\begin{array}{r l}
 & 3x^2 + 2x \\
x^2 - 5x + 6 \,\big) & 3x^4 - 13x^3 + 8x^2 + 17x - 12 \\
 & 3x^4 - 15x^3 + 18x^2 \\
\hline
 & 2x^3 - 10x^2 + 17x - 12 \\
 & 2x^3 - 10x^2 + 12x \\
\hline
 & 5x - 12 .
\end{array}
$$

$$\int 3x^2+2x+\frac{5x-12}{(x-3)(x-2)}\,dx = x^3+x^2+\int\frac{5x-12}{(x-3)(x-2)}\,dx$$

$$\frac{5x-12}{(x-3)(x-2)} = \frac{A}{x-3}+\frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)} \qquad x=3: \quad 3=A \qquad x=2: \quad -2=-B.$$

$$x^3+x^2+\int\frac{3}{x-3}+\frac{2}{x-2}\,dx = x^3+x^2+3\ln|x-3|+2\ln|x-2|+c.$$

Q19

$$\frac{x^2-5x}{(x+1)^2(x-1)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}$$

~~$x=-1$: $6=-2x$ ... -3.~~
~~$x=1$: $-4=$~~
~~$x=0$:~~

$$= \frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x+1)^2(x-1)}$$

$x=-1$: $6=-2B$ $\quad B=-3$

$x=1$: $-4=4C$ $\quad C=-1$

$x=0$: $0=-A-B+C$

$A=-B+C=2.$

$$\int\frac{2}{x+1}-\frac{3}{(x+1)^2}+\frac{-1}{x-1}\,dx = 2\ln|x+1|-\frac{3}{x+1}-\ln|x-1|+c.$$

Q20

$$\frac{4x^2+7}{(x^2+2)^2} = \frac{A}{x^2+2}+\frac{B}{(x^2+2)^2} = \frac{A(x^2+2)+B}{(x^2+2)^2} \qquad A=4,\ B=-1.$$

$$\int\frac{4}{x^2+2}-\frac{1}{(x^2+2)^2}\,dx = \int\frac{4}{x^2+2}\,dx-\int\frac{1}{(x^2+2)^2}\,dx$$

$$\int\frac{4}{x^2+2}\,dx \qquad x=\sqrt{2}\tan\theta \qquad \frac{dx}{d\theta}=\sqrt{2}\sec^2\theta \qquad \int\frac{4}{2(\tan^2\theta+1)}\frac{dx}{d\theta}\,d\theta = \int\frac{2}{\sec^2\theta}\sqrt{2}\sec^2\theta\,d\theta = \int 2\sqrt{2}\,d\theta$$

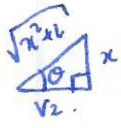
$$= 2\sqrt{2}\,\theta = 2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c.$$

$$\int\frac{1}{(x^2+2)^2}\,dx \qquad x=\sqrt{2}\tan\theta \qquad \frac{dx}{d\theta}=\sqrt{2}\sec^2\theta \qquad \int\frac{1}{\left(2(\tan^2\theta+1)\right)^2}\frac{dx}{d\theta}\,d\theta = \int\frac{1}{4\sec^4\theta}\sqrt{2}\sec^2\theta\,d\theta$$

$$= \frac{\sqrt{2}}{4}\int\cos^2\theta\,d\theta$$

$$\int\cos^2\theta\,d\theta = \int\underbrace{\cos\theta}_{u}\,\underbrace{\cos\theta}_{v'}\,d\theta = \cos\theta\sin\theta+\int\underbrace{\sin^2\theta}_{=\,1-\cos^2\theta}\,d\theta.$$

$$= \frac{\sqrt{2}}{8}\cos\theta\sin\theta+\frac{\sqrt{2}}{8}\theta+c \qquad 2\int\cos^2\theta\,d\theta = \cos\theta\sin\theta+\int 1\,d\theta = \cos\theta\sin\theta+\theta+c.$$



$$= \frac{\sqrt{2}}{8}\frac{\sqrt{2}\,x}{x^2+2}+\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c.$$

final answer:

$$2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{1}{4}\frac{x}{x^2+2}+\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c.$$