Math 232 Calculus 2 Fall 25 Spring b Part 1

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without Computer Algebra System capabilities, and a single US letter page of notes, but no phones or other assistance.
- You must show your work to receive credit for a question.

1	10	
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4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) Find $\int \frac{e^x}{(1-e^x)^3}\,dx$.

$u = 1 - e^x$

$\frac{du}{dx} = -e^x$

$$\int \frac{e^x}{u^3}\frac{dx}{du}\,du = \int \frac{e^x}{u^3}\frac{1}{-e^x}\,du = -\int u^{-3}\,du = \frac{1}{2}u^{-2} + c$$

$$= \frac{1}{2(1-e^x)^2} + c$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(2) Find $\int x^2 \ln(x)\, dx$, carefully showing which technique of integration you use.

$u = \ln(x)$ $\quad$ $v' = x^2$

$u' = \frac{1}{x}$ $\quad$ $v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^2\,dx$$

$$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + c$$

(3) Find $\displaystyle\int \frac{2-x^2}{x^2-3x+2}\,dx$.

$$\frac{2-x^2}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$$

$x=1: \quad 1 = -B$

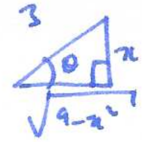
$x=2: \quad -2 = A$

$$\int \frac{-2}{x-2} + \frac{-1}{x-1}\,dx$$

$$= -2\ln|x-2| - \ln|x-1| + c$$

$$x = 3\sin\theta$$

$$\frac{dx}{d\theta} = 3\cos\theta$$



(4) Find $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx$.

$$\int \frac{x^3}{\sqrt{9-9\sin^2\theta}}\frac{dx}{d\theta}\,d\theta = \int \frac{9\sin^3\theta}{3\cos\theta}\,3\cos\theta\,d\theta = \int 9\sin\theta\,(1-\cos^2\theta)\,d\theta$$

$$u = \cos\theta$$

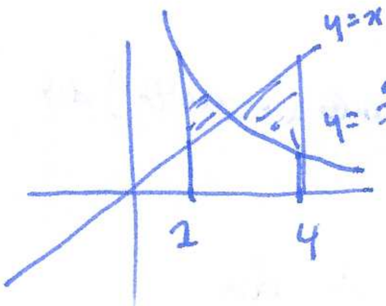
$$\frac{du}{d\theta} = -\sin\theta$$

$$\int 9\sin\theta\,(1-u^2)\frac{d\theta}{du}\,du = \int 9\sin\theta\,(1-u^2)\frac{1}{-\sin\theta}\,du$$

$$= 9\int u^2-1\,du = 3u^3 - 9u + c = 3\cos^3\theta - 9\cos\theta + c.$$

$$= 3(9-x^2)^{3/2} - 9\sqrt{9-x^2} + c$$

(5) Find the geometric area between the curves $y = x$ and $y = 4/x$ between $x = 1$ and $x = 4$.



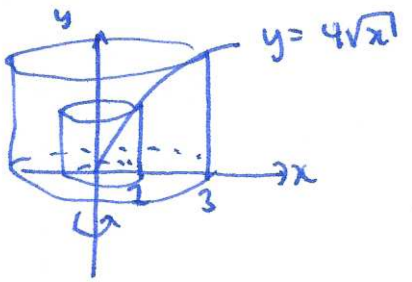
intersection: $x = \frac{4}{x} \Rightarrow x^2 = 4 \quad x = \pm 2$

$$\int_1^2 \frac{4}{x} - x \, dx + \int_2^4 x - \frac{4}{x} \, dx$$

$$= \left[4\ln|x| - \tfrac{1}{2}x^2\right]_1^2 + \left[\tfrac{1}{2}x^2 - 4\ln|x|\right]_2^4$$

$$= 4\ln(2) - 2 - \left(-\tfrac{1}{2}\right) + 2 - 4\ln(4) - \left(2 - 4\ln|2|\right).$$

(6) Find the volume of revolution obtained by rotating the region under the graph of $f(x) = 4\sqrt{x}$ about the y-axis over the interval $[1, 3]$. Sketch the solid and explicitly state the method you use. shells.



$$A = \int_1^3 2\pi r h \, dx$$

$$= \int_1^3 2\pi x 4\sqrt{x} \, dx = 8\pi \int_1^3 x^{3/2} \, dx = 8\pi \left[\frac{2x^{5/2}}{5} \right]_1^3$$

$$= \frac{16\pi}{5} \left(3^{5/2} - 1 \right)$$

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(7) Find the sum of the following infinite series:

(a) $\sum_{n=2}^{\infty} \frac{1}{2n^2 - 2n}$ $\quad \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n^2-n}$

$\frac{1}{n^2-n} = \frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1} = \frac{A(n-1)+Bn}{n(n-1)}$ $\quad n=0: \ 1 = -A$ $\quad n=1: \ 1 = B.$

$\frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n}$ $\quad S_n = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots - \frac{1}{n}\right)$ $\quad S_n = \frac{1}{2}\left(1 - \frac{1}{n}\right)$

$\lim_{n \to \infty} S_n = \frac{1}{2}$

(b) $\sum_{n=1}^{\infty} \frac{3}{4^n}$

$\frac{a}{1-r}$ $\quad \frac{3/4}{1-1/4} = \frac{3/4}{3/4} = 1$

(8) Explain whether the following series converges or diverges, indicating clearly which tests you use.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2-5}$.

limit comparison $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \left| \frac{n^{1/2}}{n^2-5} \cdot n^{3/2} \right| = \lim_{n\to\infty} \left| \frac{n^2}{n^2-5} \right| = \lim_{n\to\infty} \frac{1}{1-5/n^2} = 1 \qquad 0 < 1 < \infty$$

$\sum b_n$ converges (p-series $p>1$) $\Rightarrow$ $\sum a_n$ converges.

(b) $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}}$.

divergence test

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n}{\sqrt{n}} = \infty \neq 0.$$

(9) Explain whether the following series converges or diverges, indicating clearly which tests you use.

(a) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$.

ratio test $\lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$

converges.

(b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+3}}$.

divergence test $\lim_{n\to\infty} \frac{1}{\sqrt{1+3/n^2}} = 1 \neq 0$.

diverges.

(10) (a) Find the power series for e^{-x^2} centered at $x = 0$, and calculate it's radius of convergence.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-x)^{2n}}{n!} + \dots$

ratio test $\lim_{n\to\infty} \left| \frac{(-x)^{2n+2}}{(n+1)!} \frac{n!}{(-x)^{2n}} \right| = \lim_{n\to\infty} \frac{|x|^2}{n+1} = 0 < \infty$ converges for all x

$R = \infty$

(b) Use your answer above to find a power series for $\int e^{-x^2}\, dx$.

$\int e^{-x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{5\,2!} + \dots + \frac{(-x)^{2n+1}}{(2n+1)(n!)} + \dots$

(11) Consider the power series $\sum_{n=1}^{\infty} \frac{n(x-2)^n}{5^n}$.

(a) Find the center and the radius of convergence.

center $x=2$ ratio test $\lim_{n\to\infty} \left| \frac{(n+1)(x-2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x-2)^n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \frac{|x-2|}{5} = \frac{|x-2|}{5} < 1$

$\Rightarrow$ $|x-2| < 5$ $R = 5$.

(b) Now investigate the endpoints of the interval you got in part (a), to get the exact interval of convergence as well.

$x = 2+5$: $\sum_{n=1}^{\infty} n \frac{5^n}{5^n} = \sum_{n=1}^{\infty} n$ diverges

$x = 2-5$: $\sum_{n=1}^{\infty} n \frac{(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n$ diverges.

$f'(x) = 3x^{1/2}$

(12) Calculate the arc length of the function $f(x) = 2x^{3/2}$ over the interval $3 \leq x \leq 4$.

$\int_3^4 \sqrt{1+(f')^2}\,dx$ $\quad \int_3^4 \sqrt{1+9x}\,dx = \left[\frac{1}{9} \frac{2}{3} (1+9x)^{3/2} \right]_3^4.$

$= \frac{2}{27} \left[(1+36)^{3/2} - (1+27)^{3/2} \right].$