Math 232 Calculus 2 Fall 25 Spring a Part 1

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without Computer Algebra System capabilities, and a single US letter page of notes, but no phones or other assistance.
- You must show your work to receive credit for a question.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

$u = e^x - 1$

(1) Find $\int \frac{e^x}{(e^x-1)^3}\,dx$.

$\frac{du}{dx} = e^x$

$\int \frac{e^x}{u^3} \frac{dx}{du}\,du = \int \frac{e^x}{u^3} \frac{1}{e^x}\,du = \int u^{-3}\,du = \frac{u^{-2}}{-2} + c$

$= -\frac{1}{2} \frac{1}{(e^x-1)^2} + c$

$\int uv'dx = uv - \int u'v\,dx$

(2) Find $\int x^3 \ln(x)\, dx$, carefully showing which technique of integration you use.

$$\int x^3 \ln(x)\,dx = \frac{1}{4}x^4 \ln(x) - \int \frac{1}{4}x^3\,dx$$

$u = \ln(x) \quad u' = \frac{1}{x}$

$v' = x^3 \quad v = \frac{1}{4}x^4$

$$= \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + c$$

(3) Find $\displaystyle\int \frac{x^2-2}{x^2-3x+2}\,dx$.

$$\frac{x^2-2}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{x-1} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$$

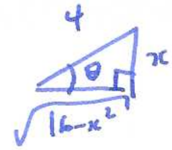
$x=1:$ $\quad -1 = -B$

$x=2:$ $\quad 2 = A$

$$\int \frac{2}{x-2} + \frac{1}{x-1}\,dx = 2\ln|x-2| + \ln|x-1| + c$$

$x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$



(4) Find $\int \frac{x^3}{\sqrt{16-x^2}}\,dx$.

$$\int \frac{x^3}{\sqrt{16-16\sin^2\theta}}\,\frac{dx}{d\theta}\,d\theta = \int \frac{64\sin^3\theta}{4\cos\theta}\,4\cos\theta\,d\theta = 64\int \sin^3\theta\,d\theta$$

$$= 64\int \sin\theta\,(1-\cos^2\theta)\,d\theta$$

$u = \cos\theta$

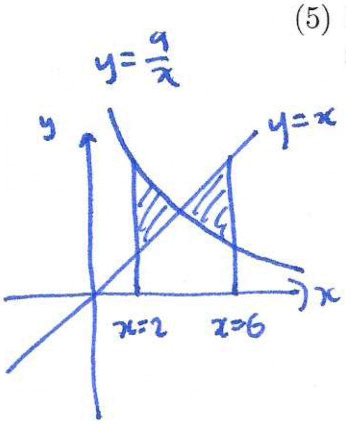
$\frac{du}{d\theta} = -\sin\theta$

$$64\int \sin\theta\,(1-u^2)\,\frac{d\theta}{du}\,du$$

$$= 64\int \sin\theta\,(1-u^2)\,\frac{1}{-\sin\theta}\,du = -64\int 1-u^2\,du = -64u + \frac{64}{3}u^3 + c$$

$$= -64\cos\theta + \frac{64}{3}\cos^3\theta + c = 64\sqrt{16-x^2} + \frac{64}{3}(16-x^2)^{3/2} + c$$

(5) Find the geometric area between the curves $y = x$ and $y = 9/x$ between $x = 2$ and $x = 6$.



intersection point : $x = \frac{9}{x}$ $x^2 = 9$ $x = \pm 3$.

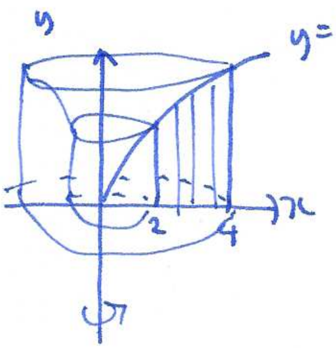
$$\int_2^3 \frac{9}{x} - x \, dx + \int_3^6 x - \frac{9}{x} \, dx$$

$$\left[9\ln(x) - \frac{x^2}{2} \right]_2^3 + \left[\frac{x^2}{2} - 9\ln(x) \right]_3^6$$

$$9\ln(3) - \frac{9}{2} - \left(9\ln(2) - 2 \right) + 18 - 9\ln(6) - \left(\frac{9}{2} - 9\ln(3) \right)$$

$$= 18\ln(3) + 7 - 9\ln(2) - 9\ln(6)$$

(6) Find the volume of revolution obtained by rotating the region under the graph of $f(x) = 2\sqrt{x}$ about the y-axis over the interval $[2, 4]$. Sketch the solid and explicitly state the method you use.



shells $A = \int_2^4 2\pi rh \, dx$

$2\pi \int_2^4 x \, 2\sqrt{x} \, dx = 4\pi \int_2^4 x^{3/2} \, dx = 4\pi \left[\frac{2x^{5/2}}{5} \right]_2^4$

$= \frac{8\pi}{5} \left(4^{5/2} - 2^{5/2} \right)$

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(7) Find the sum of the following infinite series:

(a) $\displaystyle\sum_{n=\cancel{1}2}^{\infty} \frac{1}{2n-2n^2}$

$$\frac{1}{2n-2n^2} = \frac{1}{2n(1-n)} = \frac{1}{2}\,\frac{1}{n(1-n)} = \frac{1}{2}\left(\frac{A}{n}+\frac{B}{1-n}\right) = \frac{1}{2}\,\frac{A(1-n)+Bn}{n(1-n)}$$

$n=0$: $A=1$

$n=1$: $B=1$

$$\frac{1}{2}\sum_{n=2}^{\infty} -\frac{1}{n-1}+\frac{1}{n}$$

$$S_n = \frac{1}{2}\left(-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3} \cdots\cdots +\frac{1}{n}\right)$$

$$S_n = \frac{1}{2}\left(-1+\frac{1}{n}\right) \qquad \lim_{n\to\infty} S_n = -\frac{1}{2}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5^n}$

$$\frac{a}{1-r} \qquad \frac{2/5}{1-1/5} = \frac{2/5}{4/5} = \frac{1}{2}$$

(8) Explain whether the following series converges or diverges, indicating clearly which tests you use.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 - 3}$.

Limit comparison test $b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \frac{n^{1/2}}{n^2-3} \cdot n^{3/2} = \lim_{n\to\infty} \frac{1}{1-3/n^2} = 1$ $0 < 1 < \infty$

$\sum \frac{1}{n^{3/2}}$ converges (p-series $p>1$) $\Rightarrow$ $\sum a_n$ converges.

(b) $\sum_{n=1}^{\infty} \frac{(-4)^n}{\sqrt{n}}$.

~~alternating series test~~.

divergence test:

$\lim_{n\to\infty} \frac{4^n}{\sqrt{n}} = \infty \neq 0 \Rightarrow$ diverges.

(9) Explain whether the following series converges or diverges, indicating clearly which tests you use.

(a) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$.

ratio test $\lim_{n\to\infty} \left| \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} \right| = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$

converges.

(b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+2}}$.

~~limit comparison test~~ $\cancel{b_n = \frac{1}{\sqrt{n}}}$

$\cancel{\lim_{n\to\infty} \; n.T}$

divergence test $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+2}} = 1 \neq 0$

diverges.

(10) (a) Find the power series for e^{-x^2} centered at $x = 0$, and calculate it's radius of convergence.

$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} \cdots + \frac{(-x^2)^n}{n!} + \cdots$

ratio test $\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{n+1} = 0$ converges for all x.

$R = \infty$.

(b) Use your answer above to find a power series for $\int e^{-x^2}\, dx$.

$\int e^{-x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} \cdots + \frac{(-x^2)^{n+2}}{(2n+2)n!} + \cdots$

(11) Consider the power series $\sum_{n=1}^{\infty} \frac{n(x-3)^n}{4^n}$.

(a) Find the center and the radius of convergence.

center $x=3$ radius of convergence, ratio test $\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(x-3)^n} \right|$

$= \lim_{n\to\infty} \frac{n+1}{n} \frac{|x-3|}{4} = \frac{|x-3|}{4} < 1$ $\quad |x-3| < 4$ radius of converg $R = 4$.

(b) Now investigate the endpoints of the interval you got in part (a), to get the exact interval of convergence as well.

$x = 3+4$: $\sum_{n=1}^{\infty} n \frac{(4)^n}{4^n} = \sum_{n=1}^{\infty} n$ diverges.

$x = 3-4$: $\sum_{n=1}^{\infty} n \frac{(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n$ diverges.

$$\frac{df}{dx} = 3 \cdot \frac{3}{2} x^{1/2}$$

(12) Calculate the arc length of the function $f(x) = 3x^{3/2}$ over the interval $1 \leq x \leq 3$.

$$\int_1^3 \sqrt{1+(f')^2}\,dx = \int_1^3 \sqrt{1+\cancel{\frac{9}{4}}\,\frac{81}{4}x}\,dx$$

$$\left[\frac{4}{81}\,\frac{2}{3}\left(1+\frac{81}{4}x\right)^{3/2}\right]_1^3 = \frac{8}{243}\left[\left(1+\frac{243}{4}\right)-\left(1+\frac{81}{4}\right)\right].$$