recall : chain rule : $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

substitution / change of variable

$\int f(x)\,dx$, $x(u)$ $\quad \int_{x=a}^{x=b} f(x)\,dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\frac{dx}{du}\,du$

3 things to change : function, differential, limits

mnemonic : cancelling fractions $dx = \frac{dx}{du}\,du$

Q: why does this work?

$\int f(x)\,dx$, $x(u)$ set $F(x) = \int f(x)\,dx$, so $F'(x) = f(x)$

$\int f(x)\,dx = F(x)$ differentiable wrt to u : $\frac{d}{du}(F(x(u))) = F'(x(u)).x'(u)$

$\int f(x)\,dx =$ integrate above line wrt u $= \int f(x(u))\,x'(u)\,du = \int f(x(u))\frac{dx}{du}\,du$

useful fact : $\frac{du}{dx} = 1/\frac{dx}{du}$

Examples ① $\int_0^1 e^{-7x}\,dx$ set $u = -7x$ $\frac{du}{dx} = -7$

$\int_0^{-7} e^u \frac{dx}{du}\,du = \int_0^{-7} e^u \cdot -\frac{1}{7}\,du = -\frac{1}{7}\int_0^{-7} e^u\,du = -\frac{1}{7}\left[e^u\right]_0^{-7} = -\frac{1}{7}(e^{-7}-1)$

② $\int_0^2 x^2\sqrt{x^3+1}\,dx$, $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$, $\int_0^1 \frac{x}{x^2+1}\,dx$

§ 5.8 More integrals

recall : $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ so $\int \frac{1}{x}\,dx = \ln|x| + c$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ so $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$