

Example motion under gravity, acceleration: $a(t) = x''(t) = -g$ (constant)

velocity: $v(t) = x'(t) = -gt + c$

If $v(0) = v_0$, initial velocity, then $v(t) = -gt + v_0$

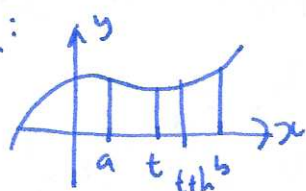
position: $x(t) = -\frac{1}{2}gt^2 + v_0t + c$

if $x(0) = x_0$, initial position, then $x(t) = -\frac{1}{2}gt^2 + v_0t + x_0$

§ 5.4 Fundamental theorem of calculus I

Thm (FTC ①) Suppose $f(x)$ is continuous on $[a, b]$ and $F(x)$ is an antiderivative for $f(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$

intuition:



consider $\int_a^t f(x) dx$

Q: what is the rate of change wrt t ?

$$\frac{d}{dt} \int_a^t f(x) dx = \lim_{h \rightarrow 0} \frac{\int_a^{t+h} f(x) dx - \int_a^t f(x) dx}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} f(x) dx$$

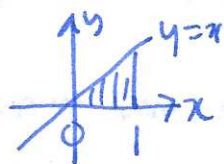
$$\approx \frac{\text{area of rectangle}}{h} = \frac{f(t) \times h}{h} = f(t)$$

i.e. $\int_a^t f(x) dx$ is an antiderivative for $f(x)$, so $\int_a^t f(x) dx = F(t) + c$

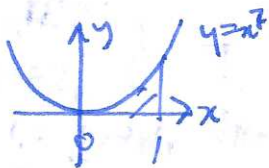
Q: what is the constant? $t = a$: $\int_a^a f(x) dx = 0 = F(a) + c \Rightarrow c = -F(a)$

$$\text{so } \int_a^t f(x) dx = F(t) - F(a) \quad \square$$

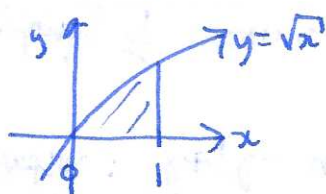
Examples



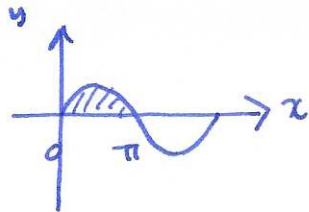
$$\int_0^1 x dx = \left[\frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$



$$\int_0^1 x^2 dx = \left[\frac{1}{3} x^3 \right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

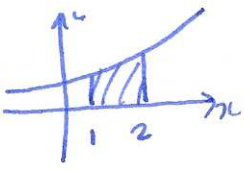


$$\int_0^1 x^{1/2} dx = \left[\frac{2}{3} x^{3/2} \right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

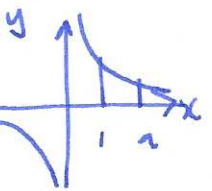


$$\int_0^{\pi} \sin(x) dx = \left[-\cos x \right]_0^{\pi} = -\cos(\pi) + \cos(0) = -(-1) + 1 = 2$$

$$\int_0^{2\pi} \sin(x) dx = \left[-\cos x \right]_0^{2\pi} = -\cos(2\pi) + \cos(0) = -1 + 1 = 0$$



$$\int_1^2 e^x dx = \left[e^x \right]_1^2 = e^2 - e$$



$$\int_1^a \frac{1}{x} dx = \left[\ln|x| \right]_1^a = \ln(a) - \ln(1) = \ln(a)$$

observations

① choice of antiderivative doesn't matter, let $F(x)$ and $F(x)+c$ be antiderivatives for $f(x)$. Then $\int_a^b f(x) dx = F(b) - F(a)$
 $= (F(b)+c) - (F(a)+c) = F(b) - F(a)$

② $\int_a^t f(x) dx$ is a function of t ! (and a, f) but not x (called a dummy variable or a bound variable) i.e. $\int_a^t f(x) dx = \int_a^t f(y) dy$. If you want a function of x write $\int_a^x f(t) dt$

§5.4 Fundamental theorem of calculus II

Thm (FTC ②) Let $f(x)$ be a continuous function on $[a, b]$, then

$A(x) = \int_a^x f(t) dt$ is an antiderivative for $f(x)$, i.e. $A'(x) = f(x) = \frac{dA}{dx}$,

so $\frac{d}{dx} \int_a^x f(t) dt = f(x)$. Furthermore $A(a) = 0$ $\square$.

Example $\frac{d}{dx} \int_0^{x^2} \cos(t) dt \leftarrow$ use chain rule!

§5.6 Substitution/change of variable

"reverse chain rule for integration".