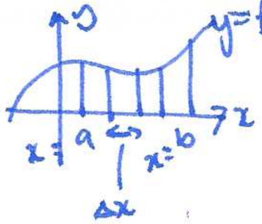
Notation



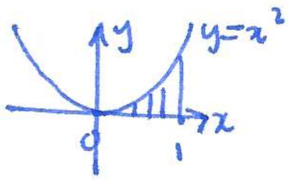
if N rectangles of equal width then $\Delta x = \frac{b-a}{N}$

left endpoint rectangles: $L_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$

right endpoint rectangles: $R_N = \sum_{i=1}^{N} f(a+i\Delta x)\Delta x$

midpoint rectangles: $M_N = \sum_{i=1}^{N} f(a+(i-\frac{1}{2})\Delta x)\Delta x$

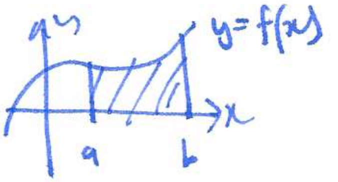
Q: what about $y=x^2$



need to find $1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

need better way...

§5.2 Definite integral



want: area under the curve $y=f(x)$ between $x=a$ and $x=b$

notation: $\int_a^b f(x)\,dx$

Formal def^n: Riemann sum $R(f, P, c)$ P: partition of $[a,b]$

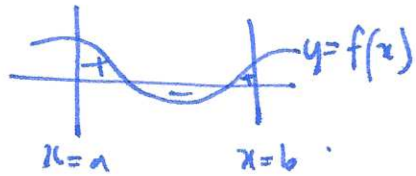
$$\int_a^b f(x)\,dx = \lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

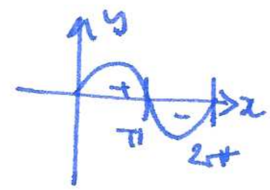
$\Delta x_i = x_i - x_{i-1}$

$c_i \in [x_{i-1}, x_i]$

note: signed area!

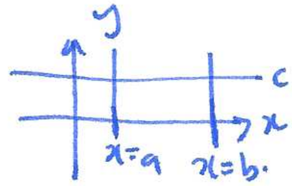


so $\int_0^{2\pi} \sin(x)\,dx = 0$



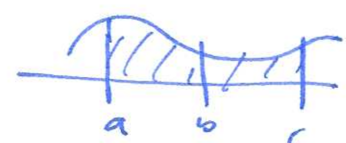
useful properties

$\int_a^b c\,dx = c(b-a)$



sums: $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

constant multiple: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$



adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

0-length interval: $\int_a^a f(x)\,dx = 0$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

comparisons: $f(x) \leq g(x)$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

§5.3 Antiderivatives

Def: A function $F(x)$ is an antiderivative for $f(x)$ if $F'(x) = f(x)$

Example $f(x) = x^2$ $F(x) = \frac{1}{3}x^3$. Note: $\frac{1}{3}x^3+1$ also works.

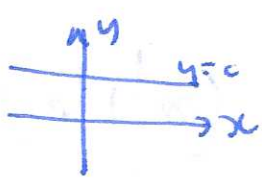
General antiderivative

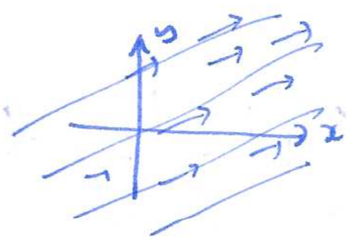
Thm let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x)+c$ for some $c \in \mathbb{R}$.

Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, then $F'(x) = G'(x) = f(x)$, so $(F(x)-G(x))' = f(x)-f(x) = 0$, so $F(x)-G(x) = c$, constant. $\square$.

Picture $f(x)$ gives the slope function for $F(x)$.

Example $f(x) = c$





$f(x) = c$

$F(x) = cx+d$.

Notation: $\int f(x)\,dx$ means general antiderivative $F(x)+c$

Example $\int x^2 + \frac{1}{x} + \sin x \, dx = \frac{1}{3}x^3 + \ln|x| - \cos x + c$

Observation: every rule for differentiation gives a rule for integration.

warning: no simple analog of product / quotient / chain rule ☹

Alternate view: we can think of finding the indefinite integral as finding a function given its slope, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general, there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$ (called an initial condition) then this picks out a particular solution.