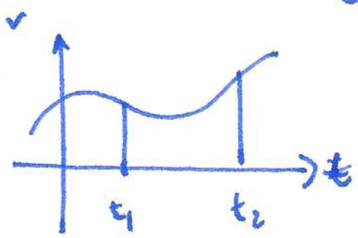
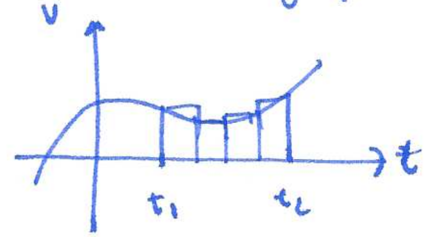
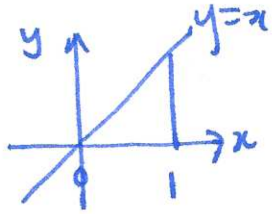
non-constant velocity: fact: distance travelled = area under the graph



finding the area: approximate by rectangles



<u>Example</u>



find area of graph under $y=x$ between $x=0$ and $x=1$ (answer $=\frac{1}{2}$)

approximate with 4 rectangles: area $\simeq$ sum of $\underbrace{\text{area of rectangles}}_{\text{width} \times \text{height}}$

$\frac{1}{4}f(0)+\frac{1}{4}f\left(\frac{1}{4}\right)+\frac{1}{4}f\left(\frac{2}{4}\right)+\frac{1}{4}f\left(\frac{3}{4}\right)$

$=\frac{1}{4}\left(f(0)+f\left(\frac{1}{4}\right)+f\left(\frac{2}{4}\right)+f\left(\frac{3}{4}\right)\right)=\frac{1}{4}\sum_{i=0}^{3}f\left(\frac{i}{4}\right)$

$=\frac{1}{4}\left(0+\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)=\frac{1}{4}\cdot\frac{6}{4}=\frac{3}{8}\simeq 0.375$

approximate with n rectangles:

$f(0)\frac{1}{n}+f\left(\frac{1}{n}\right)\cdot\frac{1}{n}+f\left(\frac{2}{n}\right)\cdot\frac{1}{n}+\dots+f\left(\frac{n-1}{n}\right)\frac{1}{n}=\sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{i}{n}\right)=\sum_{i=0}^{n-1}\frac{i}{n^2}=\frac{1}{n^2}\sum_{i=0}^{n-1}i$

$=\frac{1}{n^2}(0+1+2+\dots+n-1)$

<u>claim</u>: $1+2+3+\dots+n=\frac{1}{2}n(n+1)$

<u>proof</u> ⓪ pairing: $1+2+\dots+n-1+n$

n even: $\frac{1}{2}n(n+1)$

n odd: $\frac{n-1}{2}(n+1)+\frac{n+1}{2}=\frac{1}{2}n(n+1)$

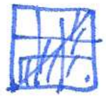
① induction: assume true for some k: $S_k=1+2+\dots+k=\frac{1}{2}k(k+1)$

$S_{k+1}=1+2+\dots+k+k+1=\frac{1}{2}k(k+1)+k+1=(k+1)\left(\frac{1}{2}k+1\right)=\frac{1}{2}(k+1)(k+2)$ ✓.

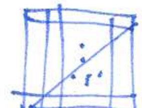
so true for $k \Rightarrow$ true for $k+1$. Check base case: $S_1=\frac{1}{2}\cdot1\cdot2=1$ ✓. □.

②

$\frac{1}{2}(2^2)+\frac{1}{2}\cdot2$



$\frac{1}{2}(3)^2+\frac{1}{2}\cdot3$



$\frac{1}{2}n^2+\frac{1}{2}n=\frac{1}{2}n(n+1)$

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so approximate area with n triangles is $\frac{1}{n^2}\frac{1}{2}(n-1)\cdot n=\frac{1}{2}\frac{n^2-n}{n^2}=\frac{1}{2}\left(1-\frac{1}{n}\right)\to\frac{1}{2}$ as $n\to\infty$.