at $x=0$ use the first derivative test:

		0	
$(x-4)$	$-$		$-$
x^3	$-$		$+$
$f'(x)$	$+$	0	$-$

$\Rightarrow$ local max.

§4.5 L'Hôpital's rule

Thm Suppose $f(x)$ and $g(x)$ are differentiable, and $f(a)=g(a)=0$ or $f(a)=g(a)=\pm\infty$

then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$, provided this limit exists.

warning ① this is not the quotient rule!

② $\dfrac{f(a)}{g(a)}$ must be indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ not $\dfrac{1}{0}$, i.e. can't use L'Hôpital on $\lim\limits_{x\to 0}\frac{1}{x}$.

Examples ① $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1}\dfrac{3x^2}{1} = 3$

② $\lim\limits_{x\to 1}\dfrac{x^{100}-1}{x-1} = \lim\limits_{x\to 1}\dfrac{100x^{99}}{1} = 100$

③ $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos^2 x}{\sin x - 1} = \lim\limits_{x\to\frac{\pi}{2}}\dfrac{-2\cos x\sin x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}} -2\sin x = -2$

④ $\lim\limits_{x\to 0+} x\ln(x) = \lim\limits_{x\to 0}\dfrac{\ln(x)}{1/x} = \lim\limits_{x\to 0+}\dfrac{1/x}{-1/x^2} = \lim\limits_{x\to 0+} -x = 0$

⑤ $\lim\limits_{x\to 0}\dfrac{e^x-x-1}{\cos x - 1} = \lim\limits_{x\to 0}\dfrac{e^x-1}{-\sin x} = \lim\limits_{x\to 0}\dfrac{e^x}{-\cos x} = -1$

⑥ $\lim\limits_{x\to 0}\dfrac{1}{\sin x}-\dfrac{1}{x} = \lim\limits_{x\to 0}\dfrac{x-\sin x}{x\sin x} = \lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin x + x\cos x} = \lim\limits_{x\to 0}\dfrac{\sin x}{\cos x+\cos x + x\sin x} = 0$

⑦ $\lim\limits_{x\to 0+} x^x = \lim\limits_{x\to 0+} e^{x\ln x}$. note e^x continuous, so $= e^{\lim_{x\to 0} x\ln x} = e^0 = 1$

Comparing growth rates of functions

Q: which grows faster, $(\ln(x))^2$ or $\sqrt{x}$?

$\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{(\ln(x))^2} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{4\ln(x)} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{2}x^{-1/2}}{4/x} = \lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{8} = \infty$

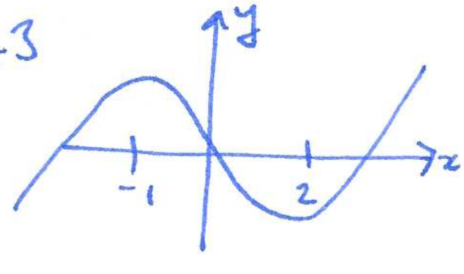
so $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial.

Proof $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$. $\square$.

§4.6 Graph sketching and asymptotes

Example sketch graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$



useful info: • critical points $f'(x) = x^2 - x - 2$
$(x-2)(x+1)$

• sign of $f'(x)$

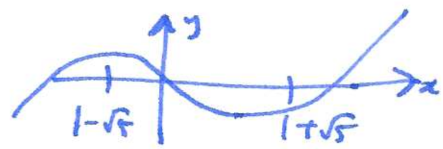
• sign of $f''(x) = 2x - 1$

critical points at $x = -1, 2$ $\quad f''(-1) = -3 < 0$ local max
$f''(2) = 3 > 0$ local min.

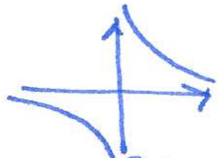
Example $f(x) = (4x - x^2)e^x$ $\qquad$ critical points $x = 1 \pm \sqrt{5}$

$f'(x) = (4 + 2x - x^2)e^x$

$f''(x) = (6 - x^2)e^x$



Asymptotes Example: $f(x) = \frac{1}{x}$



$y=0$ vertical asymptote

$x=0$ vertical asymptote.

Defn $x = c$ is a vertical asymptote if $\lim_{x\to c\pm} f(x) = \pm\infty$

$y = c$ is a horizontal asymptote if $\lim_{x\to\pm\infty} f(x) = c$

observation: rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$

Example $f(x) = \frac{x^2+x+1}{3x^2+2} \sim \frac{x^2}{3x^2} \to \frac{1}{3}$ as $x \to \infty$

Example sketch graph of $f(x) = \frac{3x+2}{2x-4}$

① find vertical asymptotes $\leftrightarrow$ denominator zero : $2x - 4 \Rightarrow x = 2$

② find $f'(x) = \frac{(2x-4).3 - (3x+2).2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2}$ $\leftarrow$ always -ve