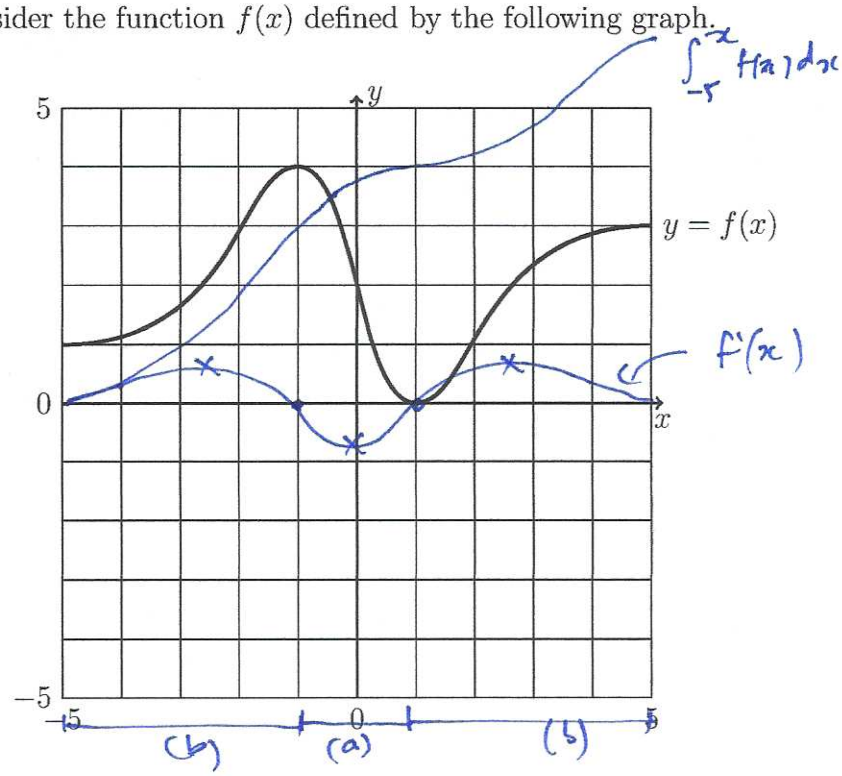
Solutions

Math 231 Calculus 1 Fall 25 Sample Midterm 3

(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $(-1,1)$

(b) Label all regions where $f'(x) > 0$. $(-5,-1) \cup (1,5)$

(c) What is $\lim_{x\to\infty} f(x)$? 3

(d) What is $\lim_{x\to-\infty} f'(x)$? 0

(e) What is $\lim_{x\to\infty} f''(x)$? 0

(f) Sketch a graph of $f'(x)$ on the figure.

(g) Sketch a graph of $\int_{-5}^{x} f(t)dt$ on the figure.

(h) Label the approximate locations of all points of inflection. $-2.5, 0, 2.5$. (x)

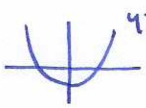
Q2 a) no vertical asymptotes b) $\lim_{x\to\infty} e^{9-x^2} = 0$, $\lim_{x\to-\infty} e^{9-x^2} = 0$

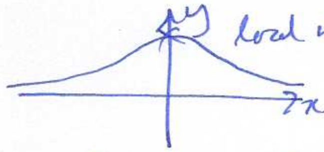
horizontal asymptote $y=0$

b) critical points, solve $f'(x)=0$: $f'(x) = e^{9-x^2}\cdot(-2x)$, critical point at $x=0$

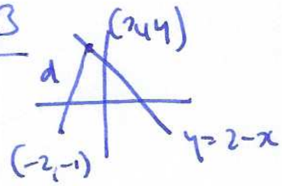
c) $f'(x)>0$ $(-\infty,0)$ $f'(x)<0$ $(0,\infty)$

d) $f''(x) = e^{9-x^2}\cdot(4x^2) - 2e^{9-x^2}$ $f''(x)=0$: $4x^2-2=0$ $x = \pm 1/\sqrt{2}$

e)  $y=4x^2-2$ concave up $(-\infty,-\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}},\infty)$ concave down $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

f) $f''(0) = -2e^9 < 0$ local max. g) 

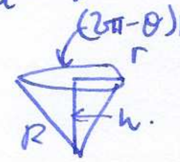
local max at $x=0$.

Q3 

$d^2 = (x+2)^2 + (y+1)^2 = (x+2)^2 + (\underbrace{2-x+1}_{3-x})^2 = \underbrace{x^2+4x+4}_{x^2-6x+9} = 2x^2-2x+13$.

$\frac{d}{dx}(d^2) = 4x-2$ critical point $x=2$, closest point $(2,0)$.

Q4 



$V = \frac{1}{3}\pi r^2 h$

$2\pi r = (2\pi-\theta)R$ set $\phi = 2\pi-\theta$

$2\pi r = \phi R$

$r = \frac{\phi R}{2\pi}$ $r^2 + h^2 = R^2$ $h^2 = R^2 - r^2 = R^2 - \left(\frac{\phi R}{2\pi}\right)^2$

$= R^2\left(1-\frac{\phi^2}{4\pi^2}\right)$.

$V = \frac{1}{3}\pi\frac{\phi^2R^2}{(2\pi)^2} R\sqrt{1-\frac{\phi^2}{4\pi^2}} = \frac{\frac{1}{3}\pi R^3}{(2\pi)^2}\phi^2\sqrt{1-\frac{\phi^2}{4\pi^2}}$.

$\frac{dV}{d\phi} = \frac{\pi R^3}{12\pi}\left(2\phi\sqrt{1-\frac{\phi^2}{4\pi^2}} + \phi^2\frac{1}{2}\left(1-\frac{\phi^2}{4\pi^2}\right)^{-1/2}\cdot\frac{-2\phi}{4\pi^2}\right)$. solve $\frac{dV}{d\phi}=0$:

$2\phi\left(1-\frac{\phi^2}{4\pi^2}\right) = \frac{\phi}{4\pi^2}\phi^2 \Rightarrow 1-\frac{\phi^2}{4\pi^2} = \frac{\phi^2}{4\pi^2}$ $\phi = \sqrt{1/2}\,\pi \cdot \frac{1}{\cancel{2}}\sqrt{2}\pi$

$\theta = 2\pi - \cancel{\frac{1}{2}}\sqrt{2}\pi$

Q5 a) $\lim_{x\to-2}\frac{x^2+x-6}{3x^2+4x-4} \overset{L'H}{=} \lim_{x\to-2}\frac{2x+1}{6x+4} = \frac{-3}{8}$

b) $\lim_{x\to0}\frac{\tan 3x}{4x} \overset{L'H}{=} \lim_{x\to0}\frac{\sec^2 3x\cdot 3}{4} = \frac{3}{4}$

c) $\lim_{x\to4}\frac{\sqrt{x}-2}{x-4} \overset{L'H}{=} \lim_{x\to4}\frac{\frac{1}{2}x^{-1/2}}{1} = \frac{1}{4}$

d) $\lim_{x\to2}\frac{x^2-4}{1/2-1/x} = \lim_{x\to2}\frac{x^3-4x}{\frac{x}{2}-1} \overset{L'H}{=} \lim_{x\to2}\frac{3x^2-4}{\frac{1}{2}} = 16$

e) $\lim_{x\to 0} \frac{\tan^{-1}(2x)}{\sin^{-1}(3x)} \overset{l'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+4x^2}\cdot 2}{\frac{1}{\sqrt{1-9x^2}}\cdot 3} = \frac{2}{3}.$

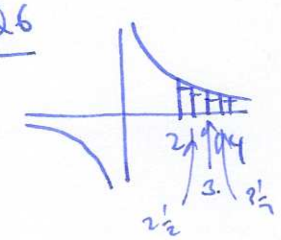
f) $\lim_{x\to\infty} \frac{x^2+1}{e^{x^{1/2}}} \overset{l'H}{=} \lim_{x\to\infty} \frac{2x}{e^{x^{1/2}}\cdot\frac12 x^{-1/2}} = \lim_{x\to\infty}\frac{4x^{3/2}}{e^{x^{1/2}}} \overset{l'H}{=} \lim_{x\to\infty} \frac{6x^{1/2}}{e^{x^{1/2}}\cdot \frac12 x^{-1/2}} = \lim_{x\to\infty} \frac{12x}{e^{x^{1/2}}}$

$\overset{l'H}{=} \lim_{x\to\infty} \frac{12}{e^{x^{1/2}}\cdot\frac12 x^{-1/2}} = \lim_{x\to\infty} \frac{24x^{1/2}}{e^{x^{1/2}}} \overset{l'H}{=} \lim_{x\to\infty} \frac{12x^{-1/2}}{e^{x^{1/2}}\cdot\frac12 x^{-1/2}} = \lim_{x\to\infty} \frac{24}{e^{x^{1/2}}} = 0$

g) $\lim_{x\to \cancel{\infty}\frac12^-} \frac{\tan \pi x}{\ln(1-2x)} \overset{l'H}{=} \lim_{x\to\frac12} \frac{\sec^2(\pi x)\cdot\pi}{\frac{1}{1-2x}\cdot -2} = \lim_{x\to\frac12} -\frac{\pi}{2}\,\frac{1-2x}{\cos^2(\pi x)} \overset{l'H}{=} \lim_{x\to\frac12} -\frac{\pi}{2}\,\frac{-2}{2\cos(\pi x)\cdot\sin(\pi x)\cdot\pi} = \infty.$

h) $\lim_{x\to\infty}\frac{2x^2-3x+4}{\sqrt{x^4-2x^2+6}} = \lim_{x\to\infty}\frac{2-3/x+4/x^2}{\sqrt{1-2/x^2+6/x^4}} = 2$

Q6



undershoot: $\frac12\left(\frac{1}{2\frac12}+\frac13+\frac{1}{3\frac12}+\frac14\right).$

Q7 a) $\int 3x^{9/5} - 2x^{4/5} + 4x^{-1/5}\,dx = \frac{15}{14}x^{14/5} - \frac{10}{9}x^{9/5} + 4\cdot\frac54 x^{4/5} + c$

b) $\int_{-2}^{3} |x|\,dx = \int_{-2}^{0} -x\,dx + \int_{0}^{3} x\,dx = \left[-\frac12 x^2\right]_{-2}^{0} + \left[\frac12 x^2\right]_{0}^{3} = 2+\frac92$

c) $\int_1^9 \frac{3}{\sqrt x}\,dx = \left[3\cdot 2x^{1/2}\right]_1^9 = 6(3-1) = 12$

d) $\int_0^2 e^{-3x}\,dx = \left[-3e^{-3x}\right]_0^2 = -3e^{-6}+3.$

e) $\int_0^x \frac{1}{t+3}\,dt = \left[\ln|t+3|\right]_0^x = \ln(x+3)-\ln(3)$

f) $\frac14\int \frac{1}{1+(x/2)^2}\,dx$ $\quad u=\frac{x}{2}$, $\frac{du}{dx}=\frac12$ $\quad \frac14\int\frac{1}{1+u^2}\cdot\frac{dx}{du}\,du = \frac14\int\frac{1}{1+u^2}\cdot 2\,du = \frac12\tan^{-1}(u)+c = \frac12\tan^{-1}\left(\frac{x}{2}\right)+c$

g) $\int\frac{x}{1+4x^2}\,dx$ $\quad u=1+4x^2$, $\frac{du}{dx}=8x$ $\quad \int\frac{x}{u}\,\frac{dx}{du}\,du = \int\frac{x}{u}\cdot\frac{1}{8x}\,du = \frac18\int\frac{1}{u}\,du = \frac18\ln|u|+c = \frac18\ln|1+4x^2|+c$

h) $\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + c$

i) $\int x\cos(2+x^2)\,dx$ $\quad u = 2+x^2 \quad \frac{du}{dx} = 2x$ $\quad \int x\cos(u)\frac{dx}{du}du = \int x\cos(u)\frac{1}{2x}du = \frac{1}{2}\int\cos(u)du$

$= \frac{1}{2}\sin(u) + c = \frac{1}{2}\sin(2+x^2) + c$

j) $\int \frac{\cos(x)}{e^{\sin x}}dx$ $\quad u = -\sin x \quad \frac{du}{dx} = -\cos x$ $\quad \int e^u\cos(x)\frac{dx}{du}du = \int e^u\cos x\frac{1}{-\cos x}du = \int -e^u du = -e^u + c$

$= -e^{-\sin x} + c$

k) $\int \frac{\cos(x)}{\sin^3(x)}dx$ $\quad u = \sin x \quad \frac{du}{dx} = \cos x$ $\quad \int \frac{\cos(x)}{u^3}\cdot\frac{1}{\cos(x)}du = \int u^{-3}du = -\frac{1}{2}u^{-2} + c = -\frac{1}{2}\frac{1}{\sin^2 x} + c$

Q8 $\quad A(x) = \int_0^x e^{-t}\sin(t)dt \quad \frac{d}{dx}(A(\sqrt{x})) = A'(\sqrt{x})\cdot\frac{1}{2}x^{-1/2}$.

$= e^{-\sqrt{x}}\sin(\sqrt{x})\cdot\frac{1}{2}x^{-1/2}$

Q9 $\quad v(t) = (2t+1)^{-3} \quad x(t) = \frac{1}{2}(2t+1)^{-2}\cdot -\frac{1}{2} + c \quad x(0) = 0 \Rightarrow c = \frac{1}{4}$.

$x(t) = \frac{1}{4} - \frac{1}{4}\frac{1}{(2t+1)^2}$ $\quad \lim_{t\to\infty} x(t) = \frac{1}{4}$, particle does not reach $x = 10$.