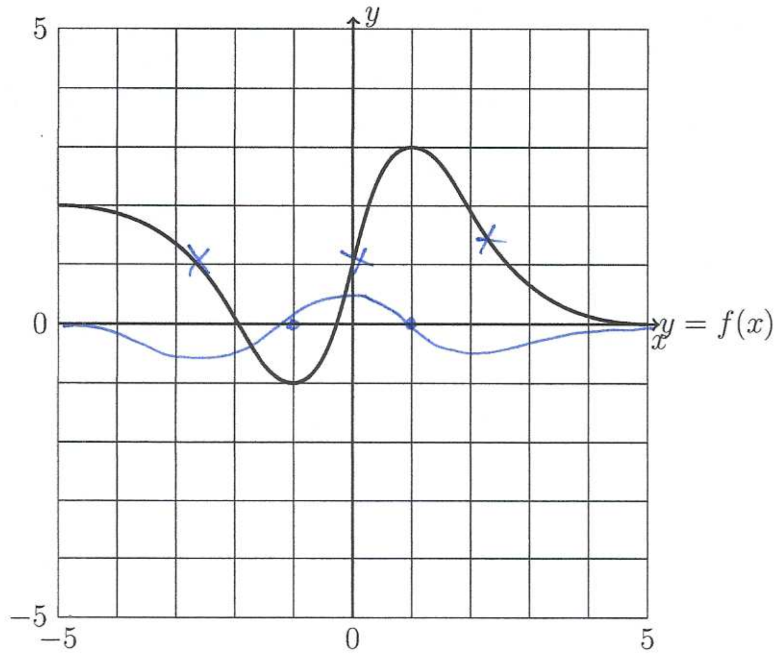
Math 231 Calculus 1 Fall 25 Sample Midterm 2

(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $(-5,-1) \cup (1,5)$
(b) Label all regions where $f'(x) > 0$. $(-1,1)$
(c) Sketch a graph of $f'(x)$ on the figure.
(d) What is $\lim_{x\to\infty} f(x)$? 0
(e) What is $\lim_{x\to-\infty} f'(x)$? 0
(f) Label the approximate locations of all points of inflection. × ·

(2) Find the derivatives of the following functions
(a) $x^3e^{-2x^4}$
(b) $\dfrac{2-\tan(3x)}{\sqrt{3x-1}}$
(c) $x^{\sqrt{x}}$
(d) $\sqrt{\csc(\ln(x))}$
(e) $\cos^{-1}(3/\sqrt[4]{x})$
(f) $\tan^{-1}(2x-3)$

(3) Find the second derivatives of the functions in parts a) and f) above.

SMT2 Solutions

Q2 a) $3x^2 e^{-2x^4} + x^3 e^{-2x^4} \cdot -8x^3$

b) $\dfrac{\sqrt{3x-1}\,(-\sec^2(3x).3) - \frac{1}{2}(3x-1)^{-1/2}.3\,(2-\tan(x))}{3x-1}$

c) $x^{\sqrt{x}} = e^{\ln(x)\,x^{1/2}}$ $\quad (x^{\sqrt{x}})' = e^{\ln(x).x^{1/2}} \cdot \left(\frac{1}{x}.x^{1/2} + \ln(x).\frac{1}{2}x^{-1/2}\right).$

d) $\frac{1}{2}(\operatorname{cosec}(\ln(x)))^{-1/2}.(-\operatorname{cosec}(\ln(x)).\cot(\ln(x)).\frac{1}{x}$

e) $\dfrac{-1}{\sqrt{1-9/\sqrt{x}}} \cdot \frac{3}{4}x^{-3/4}$

f) $\dfrac{1}{1+(2x-3)^2} \cdot 2 = 2\left(1+(2x-3)^2\right)^{-1}$

Q3 a) $6x e^{-2x^4} + 3x^2.e^{-2x^4}.(-8x^3) + 3x^2 e^{-2x^4}.-8x^3 + x^3.e^{-2x^4}(-8x^3)^2 + x^3 e^{-2x^4}.-24x^2$

f) $-2\left(1+(2x-3)^2\right)^{-2}.\left(2(2x-3).2\right)$

Q4 $4y^2+4=2x^2$

$8yy' = 4x \quad y' = \frac{x}{2y}$ at $(-2,-1)$ $\quad y'=1$ tangent line $y+1 = x+2$

$y = x+1.$

Q5 $x^2y^2 - \underset{xy^{-1}}{\frac{x}{y}} = \cos(y-x).$

$2xy^2 + x^2 2yy' - y^{-1} + x.-y^{-2}.y' = -\sin(y-x).(y'-1)$

$y'\left(x^2 2y - \frac{x}{y^2} + \sin(y-x)\right) = -2xy^2 + \frac{1}{y} + \sin(y-x)$

$y' = \dfrac{-2xy^2 + \frac{1}{y} + \sin(y-x)}{x^2 y - \frac{x}{y^2} + \sin(y-x)}$

Q6 $V = \frac{4}{3}\pi r^3 \quad A = 4\pi r^2 \qquad \frac{dA}{dt} = 8\pi r \frac{dr}{dt} \qquad \frac{dA}{dt} = 8\pi r.\frac{1}{\pi r^2} = \frac{8}{r}$

$\underset{4\,cm^3/sec}{\frac{dV}{dt}} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r^2}$

at $r = 3$ $\quad \frac{dA}{dt} = \frac{8}{3}$ cm/sec

Q7 $f(x) = \sqrt[3]{x} = x^{1/3}$ $\quad f(27) = 3$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\cdot\sqrt[3]{x^2}}$ $\quad f(26) \approx f(27) + f'(27).(-1)$

$3 + \frac{1}{3.9}\cdot -1 = 3 - \frac{1}{27}$

Q8 $f(x) = e^{-x}(x^2+3x-3)$

$f'(x) = -e^{-x}(x^2+3x-3) + e^{-x}(2x+3) = -e^{-x}(x^2+x-6)$

$f'(x) = 0 \Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3, 2$

		-3		2	
$-e^{-x}$	$-$		$-$		$-$
$(x+3)$	$-$		$+$		$+$
$(x-2)$	$-$		$-$		$+$
$f'(x)$	$-$		$+$		$-$

$x=-3$: local min

$x=2$ local max

Q9 $f'(x) = 4x - 3$ $\quad$ critical point: $x = +\frac{3}{4}$

evaluate $f(-2) = 16$ $\leftarrow$ abs max

$f(-\frac{3}{4}) = 2\cdot\frac{9}{16} - \frac{9}{4} + 2 = \frac{7}{8}$ $\leftarrow$ abs min

$f(2) = 4$

Q10 $f(x) = \frac{e^x}{4-x^2}$ $\quad$ a) vertical asymptotes $x = \pm 2$

horizontal asymptotes: $\lim_{x\to\infty} \frac{e^x}{4-x^2} = +\infty$ $\quad \lim_{x\to-\infty} \frac{e^x}{4-x^2} = 0$.

b) $f'(x) = \frac{(4-x^2)e^x - e^x(-2x)}{(4-x^2)^2} = \frac{e^x(4+2x-x^2)}{(4-x^2)^2}$ $\quad f'(x) = 0$: $x = \frac{2\pm\sqrt{4+16}}{-2} = +1\pm\sqrt{5}$.

c)

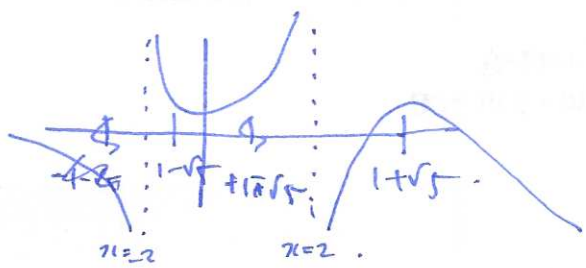
		$+1-\sqrt{5}$		$+1+\sqrt{5}$	
$f'(x)$	$-$		$+$		$-$

increasing $(+1-\sqrt{5}, +1+\sqrt{5})$ decreasing $(-\infty, +1-\sqrt{5}) \cup (+1+\sqrt{5}, \infty)$.

d) $\leftarrow$ skip 2nd derivative test, use first derivative test $\quad +1-\sqrt{5}$ local ~~max~~ min

$+1+\sqrt{5}$ local. max

e)



Q11 $f'(x)>0$ so max value on $[1,3]$ at $x=3$.

Q12 a) $\lim_{x\to\infty}\frac{\sqrt{3+4(-x)^2}}{3(-x)-2} = \lim_{x\to\infty}\frac{\sqrt{3+4x^2}}{-3x-2} = \lim_{x\to\infty}\frac{\sqrt{4+3/x^2}}{-3-2/x^2} = -\frac{2}{3}$

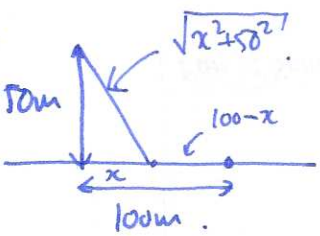
b) $\overset{l'H}{=}\lim_{x\to \cancel{\infty}\,0}\frac{\frac{1}{\sqrt{1-x^2/2}}\cdot\frac12}{\frac{-1}{\sqrt{1-36x^2}}\cdot 6} = \lim_{x\to0} -\frac{1}{12}\frac{\sqrt{1-36x^2}}{\sqrt{1-x^2/2}} = -\frac{1}{12}$

c) $\lim_{x\to0}\frac{\ln(x)}{\underbrace{1/\sin(x)}_{=\csc(x)}} \overset{l'H}{=} \lim_{x\to0} \frac{1/x}{-\csc(x)\cot(x)} = \lim_{x\to0} \frac{-\sin^2 x}{x\cos(x)} \overset{l'H}{=} \lim_{x\to0} \frac{-2\sin x\cos x}{\cos(x) \cancel{+} - x.\sin x} = 0$

d) $\lim_{x\to0}\frac{e^{4x}-1-\sin 4x}{(\sin 4x)(e^{4x}-1)} \overset{l'H}{=} \lim_{x\to0} \frac{\cancel{4}e^{4x}-\cos 4x.\cancel{4}}{\cos(4x).\cancel{4}(e^{4x}-1)+\sin 4x.e^{4x}.\cancel{4}}$

$\overset{l'H}{=}\lim_{x\to0}\frac{4e^{4x}-\sin 4x.4}{-\sin(4x).4(e^{4x}-1)+\cos(4x).4e^{4x}+\cos 4x.4e^{4x}+\sin 4x.4e^{4x}} = \frac48 = \frac12$

Q13



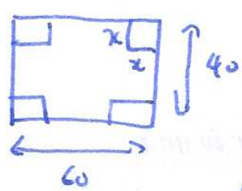
$T=\frac{\sqrt{x^2+50^2}}{v}+\frac{100-x}{4v}$

$\frac{dT}{dx} = \frac{1}{\cancel{2}v}(x^2+50^2)^{-1/2}.\cancel{2}x + -\frac{1}{4v}$

$\frac{dT}{dx}=0$: $\frac{2x}{\sqrt{x^2+50^2}}=\frac14$.

$\frac{4x^2}{x^2+50^2}=\frac1{16}$ $\quad(4-\frac1{16})x^2=\frac{50^2}{16}$ $\quad x=\frac{50}{16.(4-\frac1{16})}$.

Q14

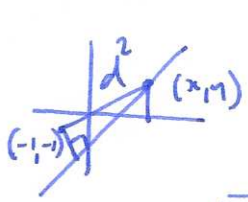


$V=x(40-2x)(60-2x)=4x(20-x)(30-x)=2400x-200x^2+4x^3$

$\frac{dV}{dx}=\cancel{(40-2x)(60-2x)}\ 2400-400x+8x^2$

critical point: $x^2-50x+\cancel{8}300=0$ $\quad x=\frac{50\pm\sqrt{50^2-1200}}{2}$.

Q15



$d^2=(x+1)^2+(y+1)^2=(x+1)^2+(3x-2+1)^2=(x+1)^2+(3x-1)^2$

$(d^2)'=2(x+1)+2(3x-1).3=2x+2+18x-6=20x-4$ $\quad x=\frac15$

$d=\sqrt{(x+\frac15)^2+(y+\frac15)^2}$.