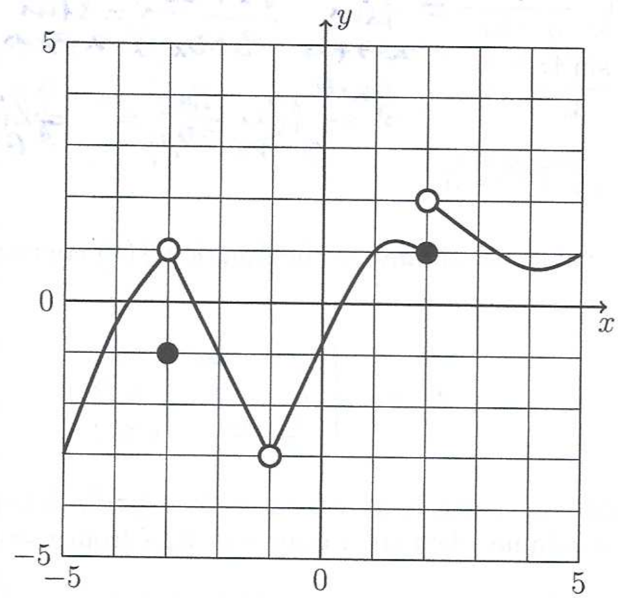
Solutions

Math 231 Calculus 1 Fall 25 Sample Midterm 1

(1) The graph of $y = f(x)$ is shown below. Evaluate each limit, or write DNE if the limit does not exist. No justifications are necessary.



(a) $\lim_{x\to 2-} f(x)$ 1

(b) $\lim_{x\to 2+} f(x)$ 2

(c) $\lim_{x\to 2} f(x)$ DNE

(d) $\lim_{x\to -1-} f(x)$ −3

(e) $\lim_{x\to -1+} f(x)$ −3

(f) $\lim_{x\to -3} f(x)$ 1

solutions 2

(2) Evaluate these limits. For an infinite limit, write $+\infty$ or $-\infty$. If a limit does not exist (DNE), you must justify why this is the case.

(a) $\lim\limits_{x\to 2}\dfrac{x-2}{x^2+x-6} = \lim\limits_{x\to 2}\dfrac{(x-2)}{(x-2)(x+3)} = \lim\limits_{x\to 2}\dfrac{1}{x+3} = \dfrac{1}{5}$

(b) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{3+2x^2}}{3-2x} = \lim\limits_{x\to+\infty}\dfrac{\sqrt{3+2x^2}/x}{3+2x/x} = \lim\limits_{x\to\infty}\dfrac{\sqrt{3/x^2+2}}{3/x+2} = \dfrac{\sqrt{2}}{2}$

(c) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x}$ $\quad 4x=\theta$ $\quad = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{3\theta/4} = \dfrac{4}{3}\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = \dfrac{4}{3}.$

(d) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+25}-5}$

(3) For what value of c (if any) is the function $f(x)$ continuous at $x=2$? Justify your answer.

$$f(x)=\begin{cases}\frac{1}{x-1}-2x & x<2\\ c & x=2\\ \frac{2\cos(\pi x)}{x} & x>2\end{cases}$$

(4) For a sphere of radius r, its volume is $V=\frac{4}{3}\pi r^3$. What is the average rate of change of volume when the radius increases from $r=4$ to $r=5$?

(5) Show that $e^x = 9-x^2$ has a solution for some $x>0$. You do not need to find this solution.

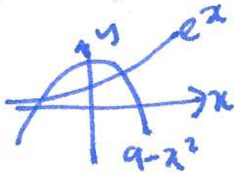
Q2 d) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{x+25}-5}\cdot\dfrac{\sqrt{x+25}+5}{\sqrt{x+25}+5} = \lim\limits_{x\to 0}\dfrac{x\sqrt{x+25}+5}{\cancel{x+25}-\cancel{25}} = \lim\limits_{x\to 0}\sqrt{x+25}+\dfrac{5}{x} = 5$

Q3 $\lim\limits_{x\to 2-}\dfrac{1}{x-1}-2x = \dfrac{1}{2}-4 = 3\frac{1}{2}$

$\neq$ so no value of c works.

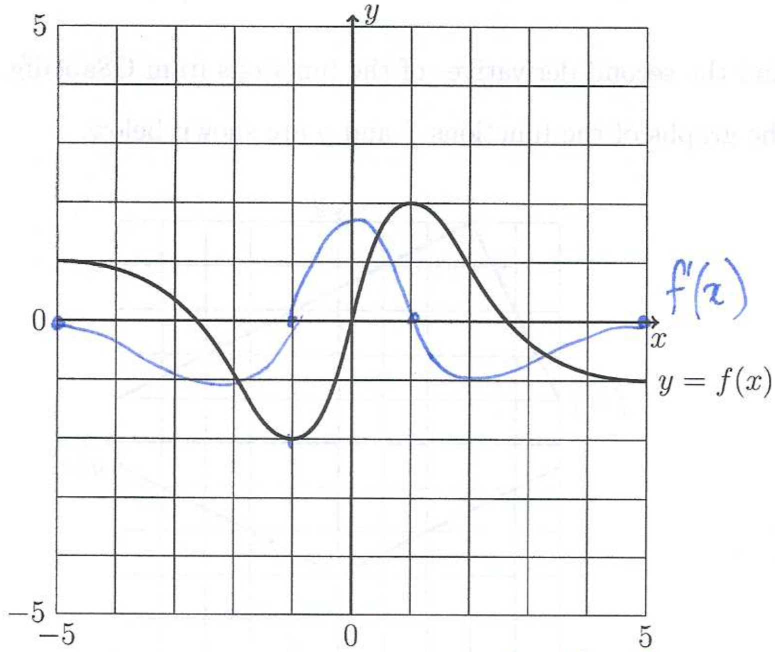
$\lim\limits_{x\to 2+}\dfrac{2\cos(\pi x)}{x} = \dfrac{-2}{2} = -1$

Q4 average rate of change: $\dfrac{V(5)-V(4)}{5-4} = \dfrac{4}{3}\pi(5^3-4^3) = \dfrac{244\pi}{3}$ $\quad 125-64$

Q5 

consider $f(x) = e^x - 9+x^2$ $\quad f(0) = -8$, $f(3) = e^3 > 0$ $\}$ IVT $\Rightarrow \exists c\in(0,3)$ s.t. $f(c)=0$.

(6) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $[-5,-1] \cup [1,5]$
(b) Label all regions where $f'(x) > 0$. $[-1,1]$
(c) Sketch a graph of $f'(x)$ on the figure.
(d) What is $\lim_{x\to\infty} f(x)$? -1
(e) What is $\lim_{x\to-\infty} f'(x)$? 0

(7) Use the limit definition of the derivative to evaluate $f'(2)$, where
(a) $f(x) = x^2 + x$
(b) $f(x) = \dfrac{1}{x+1}$
(c) $f(x) = \dfrac{1}{\sqrt{1+x}}$

(8) Find the derivatives of the following functions

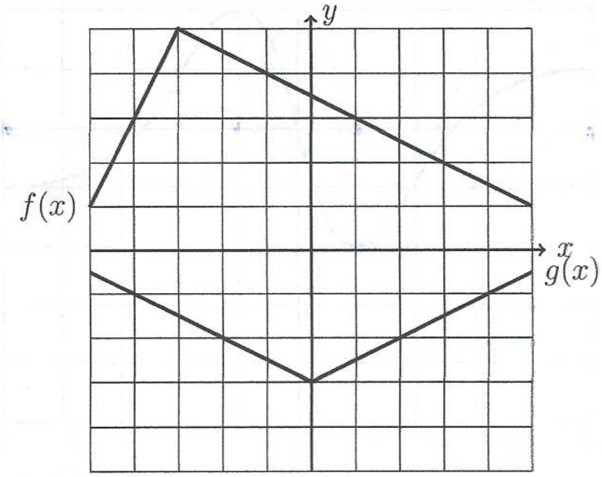
(a) $2x^4 - 3x^2$
(b) $\sqrt{x^5} - 3\sqrt[5]{1/x^2}$
(c) $4x^3e^x$
(d) $\dfrac{5x+2}{4-3x}$
(e) $\dfrac{5-\sqrt{2x}}{2-\cos(x)}$
(f) $\tan(x)$
(g) $\sin^2(x)$
(h) $x^4e^x\cos(x)$

(4)

(i) x^{x^2} (j) $\sqrt{\ln(\sec(x))}$

(9) Find the second derivatives of the functions from Q8abcdfg.

(10) The graphs of the functions f and g are shown below.



(a) Let $h(x) = f(x)g(x)$ Find $h'(3)$.
(b) Let $h(x) = f(x)/g(x)$. Find $h'(-1)$.
(c) Let $h(x) = f(g(x))$. Find $h'(2)$.

a)

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(3) = f'(3)g(3) + f(3)g'(3)$

$-\frac{1}{2} \cdot -\frac{3}{2} + 2 \cdot \frac{1}{2} = \frac{7}{4}$

b) $h'(-1) = \frac{g(\cancel{3}^{-1})f'(\cancel{3}^{-1}) - g'(\cancel{3}^{-1})f(\cancel{3}^{-1})}{(g(\cancel{3}_{-1}))^2} = \frac{(\frac{-5}{2})(-\frac{1}{2}) - (\frac{1}{2})\cdot 4}{(-\frac{5}{2})^2} = \frac{\frac{5}{4} - 2}{\frac{25}{4}} = \frac{-3}{25}$

c) $h'(x) = f'(g(x)) . g'(x)$

$h'(2) = f'(g(2))g'(2) = f'(-2) . \frac{1}{2} = -\frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4}$

Q7 a) $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(x+h)^2+(x+h)-x^2-x}{h}$

$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{x}+h-\cancel{x^2}-\cancel{x}}{h} = \lim_{h\to 0} 2x+h+1 = 2x+1$

b) $\lim_{h\to 0} \frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{\cancel{x}+\cancel{1}-(\cancel{x}+h+\cancel{1})}{(x+h+1)(x+1)} = \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} = \frac{-1}{(x+1)^2}$

c) $\lim_{h\to 0} \frac{\frac{1}{\sqrt{1+x+h}}-\frac{1}{\sqrt{1+x}}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{\sqrt{1+x}-\sqrt{1+x+h}}{\sqrt{1+x+h}\sqrt{1+x}}\cdot\frac{\sqrt{1+x}+\sqrt{1+x+h}}{\sqrt{1+x}+\sqrt{1+x+h}}$

$= \lim_{h\to 0} \frac{\cancel{1}+\cancel{x}-(\cancel{1}+\cancel{x}+h)}{h\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x}+\sqrt{1+x+h})} = \frac{-1}{(1+x)2\sqrt{1+x}}.$

Q8
Q9

a) $f(x) = 2x^4-3x^2$

$f'(x) = 8x^3-6x$

$f''(x) = 24x^2-6$

b) $f(x) = x^{5/2}-3x^{-2/5}$

$f'(x) = \frac{5}{2}x^{3/2}+\frac{6}{5}x^{-7/5}$

$f''(x) = \frac{15}{4}x^{1/2}4-\frac{42}{25}x^{-12/5}$

c) $f(x) = 4x^3e^x$

$f'(x) = 12x^2e^x+4x^3e^x$

$f''(x) = 24xe^x+\underbrace{12x^2e^x+12x^2e^x}_{24x^2e^x}+4x^3e^x$

d) $f(x) = \frac{5x+2}{4-3x}$

$f'(x) = \frac{(4-3x).5-(5x+2)(-3)}{(4-3x)^2} = \frac{35}{(4-3x)^2}.$

$f''(x) = \frac{(4-3x)^2\cdot 0-35(16-24x+9x^2)'}{(4-3x)^4} = \frac{-35(-24+18x)}{(4-3x)^4}$

e) $f(x) = \frac{5-\sqrt{2}x^{1/2}}{2-\cos(x)}$

$f'(x) = \frac{(2-\cos x)(-\sqrt{2}\cdot\frac{1}{2}x^{-1/2})-(5-\sqrt{2}x^{1/2})(\sin(x))}{(2-\cos(x))^2}$

f) $f(x) = \tan(x)$

$f'(x) = \sec^2(x)$

$f''(x) = 2\sec(x)\sec x\tan x.$

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g) $f(x) = \sin^2(x) \quad f'(x) = 2\sin x \cos x \quad f''(x) = 2\cos^2 x + -2\sin^2 x.$

h) $f(x) = x^4 e^x \cos(x)$

$f'(x) = 4x^3 e^x \cos(x) + x^4 e^x \cos(x) + x^4 e^x \cdot -\sin x$

$f''(x) = 12x^2 e^x \cos(x) + 4x^3 e^x \cos(x) + -4x^3 e^x \sin x + x^4 e^x \cos(x) - x^4 e^x \sin x - x^4 e^x \cos x.$

$+ 4x^3 e^x \cos(x) \quad - 4x^3 e^x \sin x \quad - x^4 e^x \sin x$