Math 231 Calculus 1 Fall 25 Final b Part 1

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without CAS capabilities, and a US Letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $3x^5 - \frac{2}{\sqrt{x}} + 3\sqrt[4]{x}$

$15x^4 + x^{-3/2} + \frac{3}{4}x^{-3/4}$

(b) $x^2 \ln(x)$

$2x \ln(x) + x^2 \cdot \frac{1}{x}$

$= 2x\ln(x) + x$

(2) (10 points) Find the derivative of the following functions.

(a) $\dfrac{\cos(x)}{2-e^x}$

$$\frac{(2-e^x)(-\sin x) - (-e^x)\cos(x)}{(2-e^x)^2}$$

(b) $\sin^{-1}(3x+1)$

$$\frac{1}{\sqrt{1-(3x+1)^2}} \cdot 3$$

(3) (10 points) Find the derivative of the following functions.

(a) $\sqrt{\ln(\tan(x))}$

$$\frac{1}{2}\left(\ln(\tan(x))\right)^{-1/2} \cdot \frac{1}{\tan(x)} \cdot \sec^2(x)$$

(b) $4y^3 - x^2y = e^x$ (Use implicit differentiation to find y' implicitly.)

$$12y^2y' - 2xy - x^2y' = e^x$$

$$y'(12y^2 - x^2) = e^x + 2xy$$

$$y' = \frac{e^x + 2xy}{12y^2 - x^2}$$

(4) (10 points)
(a) State the definition of $f'(x)$ as a limit.
(b) Use the limit definition of the derivative to find the derivative of $f(x) = \frac{1}{x}$. Do *not* use L'Hôpital's rule.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b) $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{x - (x+h)}{(x+h)x}$

$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$

(5) (10 points) Consider $f(x) = x^3 + 3x^2 + 3$.

(a) Find the derivative for $f(x)$, and the critical points.

critical points $f'(x) = 0$:

$f'(x) = 3x^2 + 6x$

$3x(x+2) = 0 \quad x = 0, -2$

(b) Find the equation of the tangent line at $x = 1$.

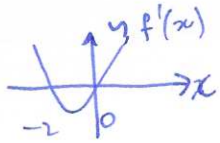
$f'(1) = 9$

$f(1) = 7$

$y - 7 = 9(x-1)$

$y = 9x - 2$

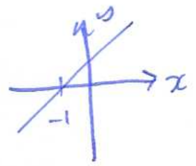
(c) Find the intervals for which $f(x)$ is decreasing.



$(-2, 0)$

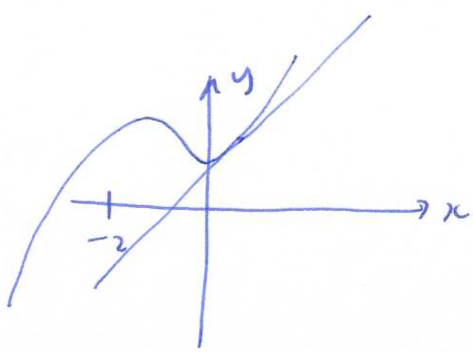
(d) Find the intervals for which $f(x)$ is concave up.

$f''(x) = 6x + 6$

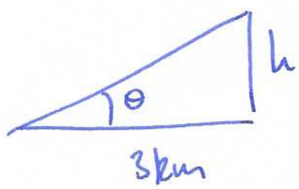


$(-1, \infty)$

(e) Sketch the graph of $f(x)$, and the tangent line at $x = 1$.



(6) (10 points) A hot air ballon rises vertically from a point on the ground 3km away. When you see it at angle $\pi/5$ radians, the rate at which the angle is increasing is 0.1 radians/sec. How fast is the balloon rising?



$$\tan\theta = \frac{h}{3}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 3\sec^2\left(\frac{\pi}{5}\right)\cdot 0.1 \approx 0.4584 \text{ km/sec}$$

(7) (10 points) Find the following limts. Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x\to 2} \frac{x^2+x-6}{3x^2-4x-4}$.

$$\overset{l'H}{=} \lim_{x\to 2} \frac{2x+1}{6x-4} = \frac{5}{8}$$

(b) Find $\lim_{x\to 0} \frac{\sin 2x}{3x}$.

$$\overset{l'H}{=} \lim_{x\to 0} \frac{2\cos 2x}{3} = \frac{2}{3}$$

(c) Find $\lim_{x\to +\infty} \frac{2x^2-3x}{e^{3x}}$.

$$\overset{l'H}{=} \lim_{x\to +\infty} \frac{4x-3}{3e^{3x}} \overset{l'H}{=} \lim_{x\to +\infty} \frac{4}{9e^{3x}} = 0$$

(8) (10 points) Evaluate the following integrals.

(a) $\displaystyle\int \left(3x^3 - 2\sqrt[3]{x} + \frac{4}{x} - \frac{2}{\sqrt{x}}\right) dx$

$$\frac{3}{4}x^4 - 2.\frac{3}{4}x^{4/3} + 4\ln|x| - 2.2.x^{1/2}$$

(b) $\displaystyle\int_0^2 e^{-2x}\, dx$

$u = -2x$

$\frac{du}{dx} = -2$

$$\int_0^{-4} e^u \frac{dx}{du}\, du = \int_0^{-4} e^u \frac{1}{-2}\, du = -\frac{1}{2}\left[e^{-4} - 1\right] \simeq 0.4908\ldots$$

(9) (10 points) Evaluate the following integrals.

(a) $\int 2x\cos(3x^2)\,dx$

$u = 3x^2$

$\frac{du}{dx} = 6x$

$\int 2x\cos(u)\frac{dx}{du}du = \int 2x\cos(u)\cdot\frac{1}{6x}du$

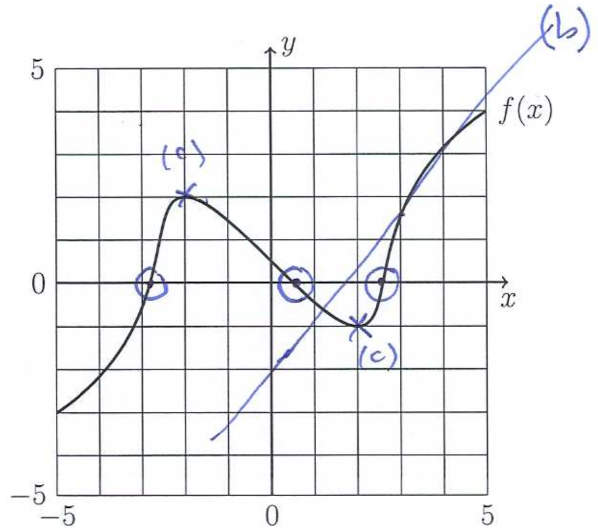
$= \frac{1}{3}\int \cos(u)du = \frac{1}{3}\sin(u)+c = \frac{1}{3}\sin(3x^2)+c$

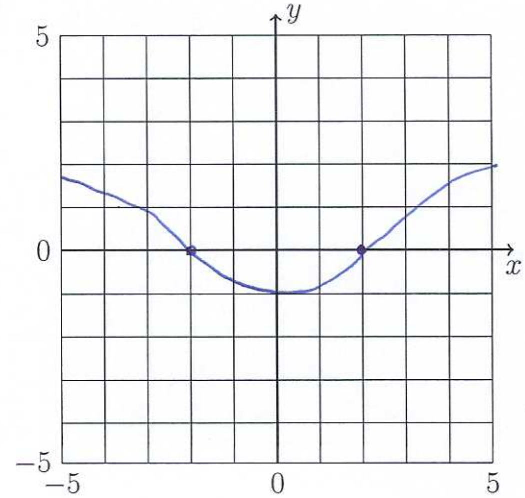
(b) If $\int_0^{12} f(x)\,dx = 3$ and $\int_8^{12} f(x)\,dx = 5$, find $\int_0^8 f(x)\,dx$.

$\int_0^8 f(x)dx + \underbrace{\int_8^{12} f(x)dx}_{5} = \underbrace{\int_0^{12} f(x)dx}_{3}$

$\int_0^8 f(x)dx = -2$

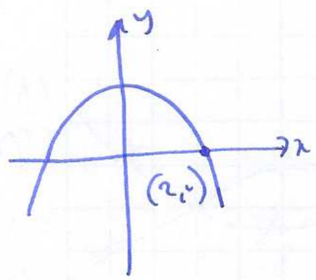
(10) (10 points) Consider the function $f(x)$ determined by the graph below.





(a) Label the roots of $f(x)$ on the graph above.
(b) On the graph above, sketch the tangent line at $x = 3$.
(c) List all the critical points of $f(x)$. $x: -2, 2$
(d) Sketch $y = f'(x)$ on the right hand graph.
(e) Estimate the intervals where $f(x)$ is concave up. $(0, 5)$

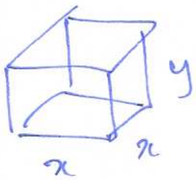
(11) (10 points) Find the area below the graph $f(x) = 8 - 2x^2$ which lies in the first quadrant.



$$\int_0^2 8 - 2x^2\,dx = \left[8x - \frac{2}{3}x^3\right]_0^2$$

$$= 16 - \frac{2}{3}8 = \frac{48 - 16}{3} = \frac{32}{3}.$$

(12) (10 points) You wish to build a metal box with a square base and top. The metal for the top and bottom costs \$1/ft^2, and the metal for the sides costs \$2ft^2. Find the dimensions of the box which minimize the cost, if the volume of the box should be 12ft^3.



$$V = 12 = x^2 y \qquad y = \frac{12}{x^2}$$

$$C = 2x^2 + 2.4xy = 2x^2 + \frac{8x.12}{x^2}$$

$$C = 2x^2 + \frac{96}{x}$$

$$\frac{dC}{dx} = 4x - \frac{96}{x^2}$$

critical point $\frac{dC}{dx} = 0$:

$$4x - \frac{96}{x^2} = 0$$

$$x^3 = 24$$

$$x = \cancel{\sqrt[3]{24}}\ \sqrt[3]{3} \cdot 2$$

$$y = \frac{12}{\cancel{24^{2/3}}\ 4 \cdot 2\sqrt[3]{3}}.$$