Math 231 Calculus 1 Fall 25 Final a Part 1

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without CAS capabilities, and a US Letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $3x^4 - 2\sqrt[3]{x} + \dfrac{3}{\sqrt{x}}$

$12x^3 - \frac{2}{3}x^{-2/3} + -\frac{3}{2}x^{-3/2}$

(b) x^2e^x

$2xe^x + x^2e^x$

(2) (10 points) Find the derivative of the following functions.

(a) $\dfrac{\sin(x)}{3-\ln(x)}$

$$\frac{(3-\ln(x)).\cos(x) - \left(-\frac{1}{x}\right)\sin x}{(3-\ln(x))^2}$$

(b) $\cos^{-1}(2x+1)$

$$\frac{-1}{\sqrt{1-(2x+1)^2}} \cdot 2$$

(3) (10 points) Find the derivative of the following functions.

(a) $\sqrt{\tan(\ln(x))}$

$$\frac{1}{2}\left(\tan(\ln(x))\right)^{-\frac{1}{2}} \cdot \sec^2(\ln(x)) \cdot \frac{1}{x}$$

(b) $3y^4 - xy^2 = e^x$ (Use implicit differentiation to find y' implicitly.)

$$12y^3y' - y^2 - x2yy' = e^x$$

$$y'(12y^3 - 2xy) = e^x + y^2$$

$$y' = \frac{e^x + y^2}{12y^3 - 2xy}$$

(4) (10 points)
(a) State the definition of $f'(x)$ as a limit.
(b) Use the limit definition of the derivative to find the derivative of $f(x) = \frac{1}{x}$. Do *not* use L'Hôpital's rule.

a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

b) $$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{x - (x+h)}{(x+h)x}$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

(5) (10 points) Consider $f(x) = x^3 + 6x^2 + 2$.

(a) Find the derivative for $f(x)$, and the critical points.

$f'(x) = 3x^2 + 12x$

critical points, solve $f'(x) = 0$

$3x(x+4) = 0 \qquad x = 0, -4$

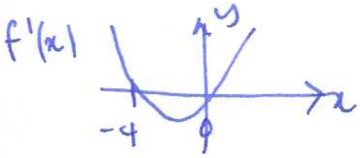
(b) Find the equation of the tangent line at $x = 1$.

$f'(1) = 15$

$f(1) = 9$

$y - 9 = 15(x - 1)$

(c) Find the intervals for which $f(x)$ is decreasing.

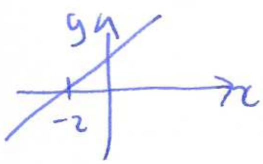


$(-4, 0)$

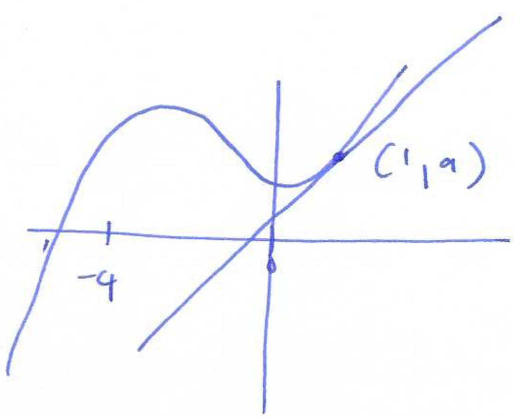
(d) Find the intervals for which $f(x)$ is concave up.

$f''(x) = 6x + 12$

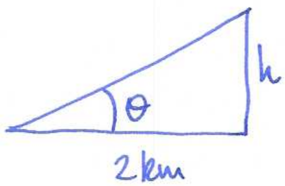
$(-2, \infty)$



(e) Sketch the graph of $f(x)$, and the tangent line at $x = 1$.



(6) (10 points) A hot air ballon rises vertically from a point on the ground 2km away. When you see it at angle $\pi/10$ radians, the rate at which the angle is increasing is 0.2 radians/sec. How fast is the balloon rising?



$$\tan\theta = \frac{h}{2}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 2\sec^2\left(\frac{\pi}{10}\right)\cdot 0.2 \approx 0.4422 \text{ km/s}$$

(7) (10 points) Find the following limts. Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x \to 3} \frac{x^2 - x - 6}{2x^2 - 7x + 3}$.

$\overset{l'H}{=} \lim_{x \to 3} \frac{2x - 1}{4x - 7} = \frac{5}{5} = 1$

(b) Find $\lim_{x \to 0} \frac{4x}{\sin 3x}$.

$\overset{l'H}{=} \lim_{x \to 0} \frac{4}{3\cos(3x)} = \frac{4}{3}$

(c) Find $\lim_{x \to +\infty} \frac{2x - 3x^2}{e^{2x}}$.

$\overset{l'H}{=} \lim_{x \to +\infty} \frac{2 - 6x}{2e^{2x}} \overset{l'H}{=} \lim_{x \to +\infty} \frac{-6}{4e^{2x}} = 0$

(8) (10 points) Evaluate the following integrals.

(a) $\int \left(2x^4 - 3\sqrt[3]{x} - \frac{2}{x} + \frac{1}{\sqrt{x}}\right) dx$

$$\frac{2}{5}x^{5} - 3.\frac{3}{4}x^{4/3} - 2\ln|x| + 2x^{1/2} + c$$

(b) $\int_0^2 e^{-3x}\, dx$

$$u = -3x$$

$$\frac{du}{dx} = -3$$

$$\int_0^{-6} e^{u}\frac{dx}{du}\,du = -\frac{1}{3}\left[e^{u}\right]_0^{-6} = -\frac{1}{3}\left(e^{-6} - 1\right)$$

$$\approx 0.3325$$

(9) (10 points) Evaluate the following integrals.

(a) $\int 3x\cos(4x^2)\,dx$

$u = 4x^2$

$\frac{du}{dx} = 8x$ $\quad \int 3x\cos(u)\frac{dx}{du}\,du = \int 3x\cos(u)\frac{1}{8x}\,du$

$= \frac{3}{8}\int \cos(u)\,du = \frac{3}{8}\sin(u) + C = \frac{3}{8}\sin(4x^2) + C$

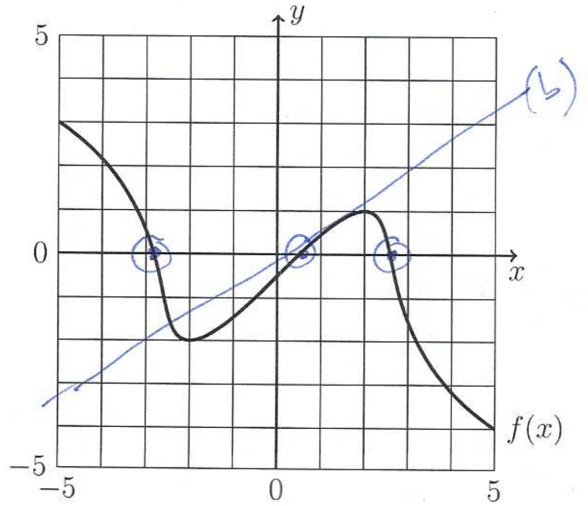
(b) If $\int_0^8 f(x)\,dx = 7$ and $\int_4^8 f(x)\,dx = 3$, find $\int_0^4 f(x)\,dx$.

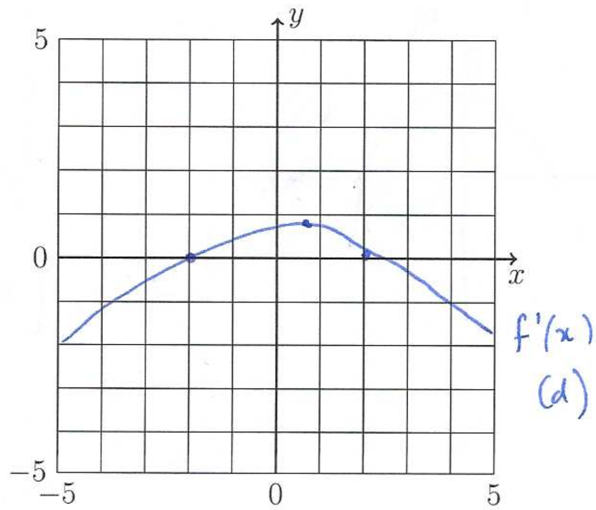
$\int_0^4 f(x)\,dx + \int_4^8 f(x)\,dx = \int_0^8 f(x)\,dx$

$3 \quad 7$

$\int_0^4 f(x)\,dx = 4$

(10) (10 points) Consider the function $f(x)$ determined by the graph below.





(a) Label the roots of $f(x)$ on the graph above.

(b) On the graph above, sketch the tangent line at $x = 1$.

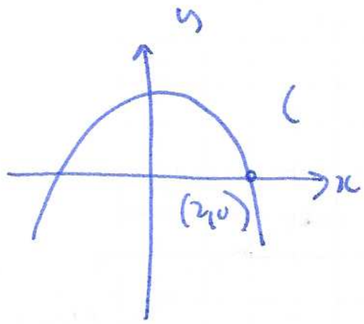
(c) List all the critical points of $f(x)$. $-2, +2$

(d) Sketch $y = f'(x)$ on the right hand graph.

(e) Estimate the intervals where $f(x)$ is concave down. $\left(\frac{1}{2}, \infty\right)$

(11) (10 points) Find the area below the graph $f(x) = 12 - 3x^2$ which lies in the first quadrant.

$= 3(4-x^2)$



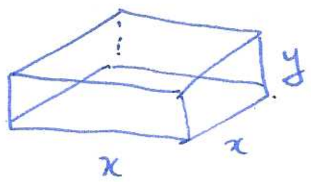
$$\int_0^2 12-3x^2\,dx$$

$$= \left[12x - x^3\right]_0^2$$

$$= 24-8 = 16$$

(12) (10 points) You wish to build a metal box with a square base and top. The metal for the top and bottom costs \$2/ft^2, and the metal for the sides costs \$1ft^2. Find the dimensions of the box which minimize the cost, if the volume of the box should be 24ft^3.

$$V = 24 = x^2 y \Rightarrow y = \frac{24}{x^2}$$



$$C = 2x^2 \cdot 2 + 4xy$$

$$C = 4x^2 + 4x \cdot \frac{24}{x^2} = 4x^2 + \frac{96}{x}$$

$$\frac{dC}{dx} = 8x - \frac{96}{x^2}$$

critical pt : $\frac{dC}{dx} = 0$:

$$8x^3 = 96$$

$$x^3 = 12$$

$$x = \sqrt{12} = 2\sqrt{3}$$

$$y = \frac{24}{12} = 2$$