Sample final solutions

Q1 a) $f(x) = 2x^3 - 3x^{-3/4} + \operatorname{cosec}(x)$

$f'(x) = 6x^2 + \frac{9}{4}x^{-7/4} - \operatorname{cosec}(x)\cot(x)$

b) $f(x) = \frac{x-x^2}{\ln(3x+2)}$ $\quad f'(x) = \frac{\ln(3x+2)(1-2x) - (x-x^2)\frac{1}{3x+2}\cdot 3}{(\ln(3x+2))^2}$

c) $f(x) = e^{-3x}\sin(2-3x)$ $\quad f'(x) = -e^{-3x}\cdot 3\sin(2-3x) + e^{-3x}\cos(2-3x).(-3)$.

d) $f(x) = (e^{-\sin(2x)}+3)^{1/4}$ $\quad f'(x) = \frac{1}{4}(e^{-\sin(2x)}+3)^{-3/4}.(e^{-\sin(2x)}.-\cos(2x).2)$

Q2 a) $\int \cancel{3}\,3x^{-4} - 2\sin x + e^x\,dx = -x^{-3} + 2\cos x + e^x + c$

b) $\int \frac{4-12x+9x^2}{x^{5/2}}dx = \int 5x^{-5/2} - 12x^{-3/2} + 9x^{-1/2}dx = -\frac{10}{7}x^{-3/2} + 12(+2)x^{-1/2} + 9x^{1/2}.2 + c$

c) $\int_0^{\pi/6}\sin^3(2x)\cos(2x)\,dx$ $\quad u = \sin 2x$, $\frac{du}{dx} = \cos 2x.2$ $\quad \int_0^{\sqrt{3}/2} u^3\cos(2x)\frac{dx}{du}du = \int_0^{\sqrt{3}/2} u^3\frac{\cancel{\cos(2x)}}{2\cos(2x)}du$

$= \frac{1}{2}\left[\frac{1}{4}u^4\right]_0^{\sqrt{3}/2} = \frac{1}{8}\left(\frac{9}{16}\right)$

d) $\int\frac{1}{4+x^2}dx = \frac{1}{4}\int\frac{1}{1+(\frac{x}{2})^2}dx$ $\quad u = \frac{x}{2}$, $\frac{du}{dx} = \frac{1}{2}$ $\quad \frac{1}{4}\int\frac{1}{1+u^2}\frac{dx}{du}du = \frac{1}{2}\int\frac{1}{1+u^2}du$

$= \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

Q3 a) $\lim_{x\to-3}\frac{x+3}{x^2+x-6} \overset{l'H}{=} \lim_{x\to-3}\frac{1}{2x+1} = \frac{1}{-5}$

b) $\lim_{x\to0}\frac{e^{2x}-1}{\sin(3x)} \overset{l'H}{=} \lim_{x\to0}\frac{2e^{2x}}{\cos(3x).3} = \frac{2}{3}$

c) $\lim_{x\to0+} e^{\ln(x)\sin(3x)} = e^{\lim_{x\to0+}\frac{\ln(x)}{1/\sin(3x)}}\Big\} \overset{l'H}{=} \lim_{x\to0+}\frac{1/x}{-(\sin(3x))^{-2}.\cos(3x).3}$

$= \lim_{x\to0+}\frac{-\cancel{3}\sin^2(3x)}{3x\cos(3x)} \overset{l'H}{=} \lim_{x\to0+}\frac{2\sin(3x).\cos(3x).3}{3\cos(3x) + 3x.-\sin(3x).3} = 0$ so $\lim_{x\to0+} x^{\sin(3x)} = e^0 = 1$

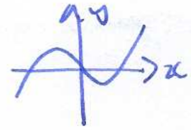
d) $\lim_{x\to0}\frac{\sin^2x - x^2}{x^2\sin^2x} \overset{l'H}{=} \lim_{x\to0}\frac{2\sin x\cos x - 2x}{2x\sin^2x + x^2.2\sin x\cos x}$ $\cancel{\overset{l'H}{=}\lim_{x\to0}\frac{2\cos^2x - 2\sin^2x - 2}{2\sin^2x + 2x\,2\sin x\cos x + 2x.2\sin x\cos x + x^2 2\cos^2x - x^2 2\sin^2x}}$

$\neq 0$ have to apply l'H 2e times -- get $-\frac{1}{3}$, too hard ~~simplify this~~

Q4 $f(x) = x^3 - 12x$ a) $f'(x) = 3x^2 - 12$ $f'(x) = 0$: $x = \pm 2$

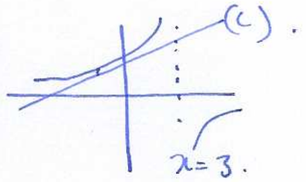
b) increasing $(-\infty, -2) \cup (2, \infty)$ c) $f''(x) = 6x$ concave up $(0, \infty)$
down $(-\infty, 0)$

d) $+2$ local min, -2 local max e)



Q6 $f(x) = \frac{3}{3-x}$. a)

$= 3(3-x)^{-1}$



b) $f'(x) = -3(3-x)^{-2} \cdot (-1)$

$f'(-1) = \frac{3}{4^2} = \frac{3}{16}$ $y - \frac{3}{4} = \frac{3}{16}(x+1)$

$f(-1) = \frac{3}{4}$

Q7 $f(x) = \frac{1}{x} - x$ $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - (x+h) - \frac{1}{x} + x}{h}$

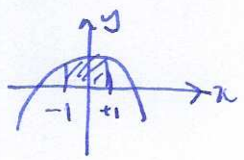
$= \lim_{h \to 0} \frac{1}{h}\left(\frac{x - (x+h)}{x(x+h)} - h\right) = \lim_{h \to 0} \frac{-1}{x(x+h)} - 1 = \frac{-1}{x^2} - 1$

Q8 $x^2y^2 + 3x - 2y = 5$.

$2xy^2 + x^2 2yy' + 3 - 2y' = 0$ at $(-2, -1)$: $-4 + -8y' + 3 - 2y' = 0$ $y' = \frac{-1}{10}$

$y + 1 = \frac{-1}{10}(x+2)$

Q10



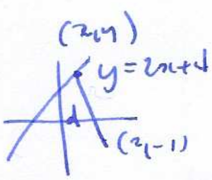
$\int_{-1}^{1} 9 - x^2 \, dx = \left[9x - \frac{1}{3}x^3\right]_{-1}^{1} = 9 - \frac{1}{3} - \left(-9 + \frac{1}{3}\right) = 18 - \frac{2}{3}$

Q11 $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $2 = 4\pi 6^2 \frac{dr}{dt}$ $\frac{dr}{dt} = \frac{1}{72}$ in/sec.

Q12 $f(x) = x^{1/3}$ $f(27) = 3$.

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$ $f(26) \approx f(27) - f'(27) = 3 - \frac{1}{3 \cdot 9} = 3 - \frac{1}{27}$.

Q13



$d^2 = (x-2)^2 + (y+1)^2$ $d^2 = (x-2)^2 + ((2x+4)+1)^2$.

$\frac{d}{dx}(d^2) = 2(x-2) + 2(2x+5) \cdot 2 = 16x - 16$ critical pt $x = 8/5$

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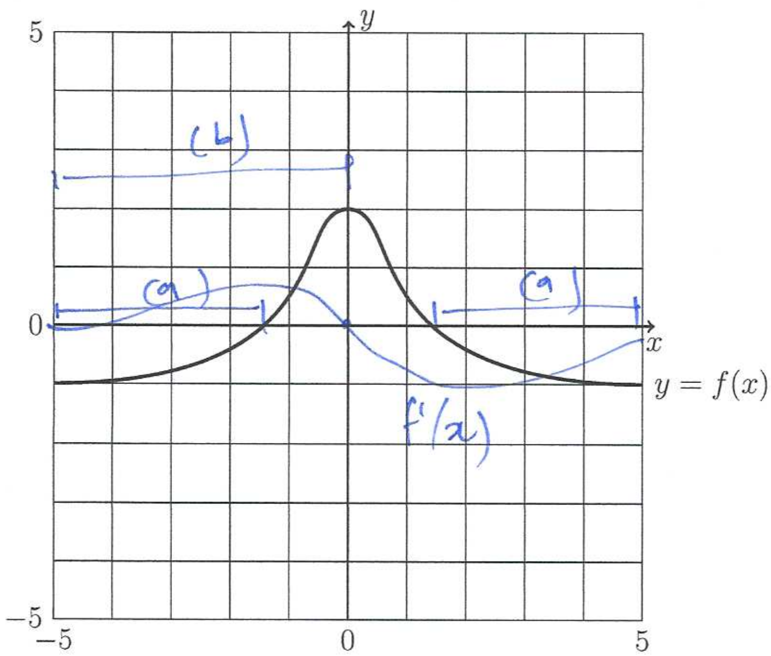
(c) Give the intervals for which f is concave up, and for which it is concave down.
(d) Decide which critical points are maxima, minima, or neither.
(e) Sketch the graph of $f(x)$.

(5) Consider the function $f(x)$ defined by the following graph.



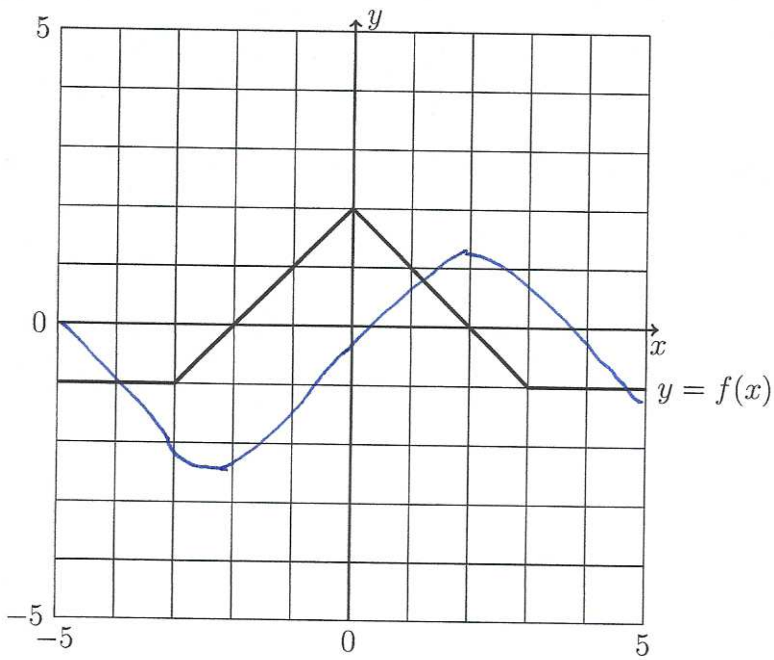
(a) Label all regions where $f(x) < 0$.
(b) Label all regions where $f'(x) > 0$.
(c) Sketch a graph of $f'(x)$ on the figure.

(6) Consider $f(x) = \dfrac{3}{3-x}$.
(a) Sketch the graph of $f(x)$ showing any asymptotes.
(b) Find the slope of the tangent line at $x = -1$, and write down the equation for the tangent line.
(c) Sketch the tangent line at $x = -1$ on your graph.

(7) Let $f(x) = \frac{1}{x} - x$. Find the derivative *using the limit definition of the derivative.* Do not use L'Hôpital's rule. Show all your work.

(8) Use implicit differentiation to find the tangent line to the curve given by the equation $x^2y^2 + 3x - 2y = 5$ at the point $(-2, -1)$.

(9) Sketch the graph of $\int_{-5}^{x} f(t)dt$, where $f(x)$ is shown below.



(10) A region in the plane is bounded by the x-axis, the graph $y = 9 - x^2$, and the lines $x = -1$ and $x = 1$.
  (a) Sketch the region (shading it in) and label the boundaries.
  (b) Find the area of the region.

(11) You blow up a spherical balloon at the rate of $2\text{in}^3/\text{s}$. How fast is the volume growing when $r = 6\text{in}$? (The volume of a sphere is $V = \frac{4}{3}\pi r^3$.)

(12) Use linear approximation to estimate $\sqrt[3]{26}$. Use you calculator to find the exact value, and find the absolute and percentage errors.

(13) What's the closest point on the line $y = 2x + 4$ to the point $(2, -1)$?