Math 301 Introduction to Proof Fall 24 Midterm 2

Name: ______________________________

(1) (a) Prove that for all integers n, if n^2 is even, then n is even.
   (b) Prove that $\sqrt{2}$ is irrational. (You may use part (a).)

(2) Give examples of:
   (a) a function $f\colon \mathbb{R} \to \mathbb{R}$ which is surjective but not injective.
   (b) a function $f\colon \mathbb{Z} \to \mathbb{N}$ which is injective but not surjective.

(3) What do the following statements mean in ordinary language?
   (a) $(\exists\ x \in \mathbb{R})(\forall\ a \in A \subseteq \mathbb{R})(x > a)$
   (b) $(\exists\ b \in \mathbb{N})(b|a \text{ and } 1 < b < a)$
   (c) $(\exists\ A \subseteq \mathbb{R})(\forall\ x \in \mathbb{R})(\exists\ a \in A)(x > a)$

(4) State the negation of the following statements, using appropriate quantifiers:
   (a) The function $f\colon \mathbb{R} \to \mathbb{R}$ is strictly decreasing.
   (b) The set $A \subseteq \mathbb{R}$ is bounded above.

(5) Write out a careful proof or give a counterexample to the following statement:
   Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be increasing functions. Then $f \circ g$ is increasing.

(6) Write out a careful proof or give a counterexample to the following statement:
   Let $f\colon X \to Y$ be a function, and let $A, B \subseteq X$. Then $f(A) \cap f(B) \subseteq f(A \cap B)$.

(7) Write out a careful proof or give a counterexample to the following statement:
   If $f \circ g$ is surjective then f is surjective.

(8) Write out a careful proof or give a counterexample to the following statement:
   If $f \circ g$ is injective then g is injective.

(9) Consider the statement:
   If $f\colon \mathbb{R} \to \mathbb{R}$ is strictly decreasing then it is injective.
   (a) State the contrapositive of the statement, and then prove or give a counterexample.
   (b) State the converse of the statement, and then prove or give a counterexample.

(10) Consider the statements $(\forall\ x \in U)(p(x) \text{ or } q(x))$ and $(\exists\ x \in U)(p(x) \text{ and } q(x))$.
   (a) What do these statements mean in terms of the truth sets of $p(x)$ and $q(x)$?
   (b) What is the logical relation between the statements, i.e, are they equivalent, does one imply the other, or are they inequivalent?
   (c) Justify your answer above. In particular, if the statements are not equivalent, give specific examples of statements $p(x)$ and $q(x)$ that show this.

MT2 Solutions

Q1 a) Thm If n^2 is even, then n is even. Proof (contrapositive) Assume n is odd, then $n=2k+1$ for some $k \in \mathbb{Z}$, which implies $n^2 = (2k+1)^2 = 4k^2+4k+1$, which is odd, as required $\square$.

b) Thm $\sqrt{2}$ is irrational. Proof (by contradiction) Assume $\sqrt{2} = a/b$ in lowest terms. Then $2 = a^2/b^2$ so $a^2 = 2b^2$, so a^2 is even. By 1a) a is even, so $a = 2c$ for some $c \in \mathbb{Z}$. Then $(2c)^2 = 4c^2 = 2b^2 \Rightarrow 2c^2 = b^2 \Rightarrow b^2$ is even. Again by 1a) b is even # a, b coprime. $\square$.

Q2 a) $f(x) = x(x-1)(x+1)$ b) $f(x) = \begin{cases} 3x+1 & x \geq 0 \\ -3x+2 & x < 0 \end{cases}$

Q3 a) A is bounded ~~below~~ above b) a is not prime c) There is a subset of $\mathbb{R}$ not bounded below.

Q4 a) strictly decreasing: $(\forall x,y \in \mathbb{R})(x<y \Rightarrow f(x) > f(y))$

negation: $(\exists x,y \in \mathbb{R})(x<y \text{ and } f(x) \leq f(y))$.

b) $A \subseteq \mathbb{R}$ bounded above: $(\exists x \in \mathbb{R})(\forall a \in A)(a < x)$.

negation: $(\forall x \in \mathbb{R})(\exists a \in A)(a \geq x)$

Q5 Thm Let $f:\mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be increasing. Then $f \circ g$ is increasing

Proof Increasing means: for all $x,y \in \mathbb{R}$ $x<y \Rightarrow f(x) \leq f(y)$

and $x<y \Rightarrow g(x) \leq g(y)$.

Suppose $x<y$ then as f increasing $g(x) \leq g(y)$. As f increasing $f(g(x)) \leq f(g(y))$ so $f \circ g$ is increasing, as required. $\square$.

Q6 counterexample

$$\begin{tikzcd} 1 \arrow[dr, "f"] & \\ & 3 \\ 2 \arrow[ur] & \end{tikzcd}$$

$A = \{1\}$, $B = \{2\}$.

Q7 ~~counterexample~~ Thm If $f \circ g$ surjective then f surjective.

$g: X \to Y$ $f: Y \to Z$ Proof Suppose $f \circ g$ is surjective. Then for any $z \in Z$, there is an $x \in X$ s.t. $f(g(x)) = z$. But then there is a $y = g(x) \in Y$ s.t. $f(y) = f(g(x)) = z$, so f is surjective. $\square$.

Q8 $X \xrightarrow{g} Y \xrightarrow{f} Z$ Thm If $f \circ g$ is injective, then g is injective Proof (contrapositive) Suppose g is not injective, then there are $a, b \in X$ s.t. $a \neq b$ but $g(a) = g(b)$. But then $f(g(a)) = f(g(b))$ so $f \circ g$ not injective. $\square$.

Q9 a) Thm If $f: \mathbb{R} \to \mathbb{R}$ is not injective, then it is not strictly decreasing.

Q9 a) <u>Proof</u> Suppose $f:\mathbb{R}\to\mathbb{R}$ is not injective, then there is $x,y\in\mathbb{R}$ with $x\neq y$ s.t. ②
$f(x)=f(y)$. If $x\neq y$, up to relabeling $x<y$ and $f(x)=f(y)$, so f not strictly decreasing. $\square$.

b) If f is ~~not~~ injective, then f is strictly decreasing. Counterexample: $f(x)=x$.

Q10 a) $(\forall x\in U)(\ p(x) \text{ or } q(x)\ ) \overset{①}{\leftrightarrow} T_p\cup T_q=U$. $(\exists x\in U)(\,p(x) \text{ and } q(x)\,)$.
truth set of $p: T_p\subseteq U$ ② $\updownarrow$
$q: T_q\subseteq U$ $T_p\cap T_q\neq\phi$.

b) inequivalent, i.e. can have $T_p\cup T_q=U$ but $T_p\cap T_q=\phi$.
$U=\mathbb{Z}$
e.g. $p(x)=x$ is odd, $q(x)=x$ is even.

and can have $T_p\cup T_q\neq U$ but $T_p\cap T_q \neq\phi$.

e.g. $U=\mathbb{Z}$ $p(x): x\in 2\mathbb{Z}$ $q(x): x\in 3\mathbb{Z}$.