Math 301 Introduction to Proof Fall 24 Sample Midterm 2

(1) Show that $\sqrt[3]{3}$ is irrational.

(2) Find an example of a function $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ which is
  (a) injective but not surjective
  (b) neither injective nor surjective

(3) Find an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is
  (a) surjective but not injective
  (b) neither injective nor surjective

(4) Find explicit bijections between the following subintervals of $\mathbb{R}$.
  (a) $(0, 1)$ and $(1, \infty)$
  (b) $(1, \infty)$ and $(0, \infty)$
  (c) $(0, \infty)$ and $\mathbb{R}$
  (d) $(0, 1)$ and $\mathbb{R}$

(5) Say a set A is *countable* if there is an injective map $f\colon A \to \mathbb{N}$. Show that the product of countable sets is countable. Show that the set of quadratic integers is countable. (A number is a quadratic integer if it is the solution of a quadratic equation with integer coefficients and leading coefficient equal to 1.)

(6) What do the following statements mean in ordinary language? Write out their negations using quantifiers.
  (a) $(\forall n \in \mathbb{N})\ (x \neq 3n)$
  (b) $(\exists a, b \in \mathbb{N})\ (\pi = a/b)$
  (c) $(\forall a, b \in A)\ \ (f(a) = f(b) \Rightarrow a = b)$, assume $f\colon A \to B$
  (d) $(\forall n \in \mathbb{N})\ (\exists m \in \mathbb{N})\ (n^2 + n = 2m)$
  (e) $(\forall A \subset \mathbb{R})\ (\exists x \in \mathbb{R})\ (\forall a \in A)(x > a)$
  (f) $(\forall n \in \mathbb{N})\ (\exists p \in \mathbb{N})\ ((p > n)$ and $[(\forall q \in \mathbb{N})\ (q|p \Rightarrow (q = 1)$ or q = p)])

(7) State the negation of the following statements, using appropriate quantifiers.
  (a) $\sqrt{\pi}$ is irrational.
  (b) The function f is injective but not surjective.
  (c) The integer n is divisible by two distinct primes.
  (d) There is an injective function $f\colon A \to B$.
  (e) The function $f\colon \mathbb{R} \to \mathbb{R}$ is increasing.
  (f) $\lim_{x \to 2} f(x) = 1$.

(8) Consider the statement: If $\mathcal{P}(\mathcal{S}) = \varnothing$ then $S = \varnothing$.
   (a) Is the statement true or false?
   (b) State the contrapositive. Is it true or false?
   (c) State the converse. Is it true or false?
   (d) State the contrapositive of the converse. Is it true or false?
   (e) Consider the statement: If $S = \varnothing$ then $\mathcal{P}(\mathcal{S}) \neq \varnothing$. Is this statement true or false? Is this statement equivalent to any of the statements above?

(9) Write out careful proofs, or give counterexamples, to the following statements.
   (a) Show that the sum of the squares of any two consecutive integers is odd.
   (b) Show that the sum of any four consecutive integers is divisible by 4.
   (c) A function $f\colon A \to B$ has an inverse if and only if it is both surjective and injective.
   (d) If f is not injective, then it is surjective.
   (e) If f is injective, then $f \circ g$ is injective.
   (f) If f and $f \circ g$ are surjective, then f is surjective.
   (g) If $f \circ g\colon \mathbb{R} \to \mathbb{R}$ is increasing, and $f\colon \mathbb{R} \to \mathbb{R}$ is increasing, then g is increasing.
   (h) If $x \in \mathbb{R}$ and $x^2 \geqslant x$, then $x \geqslant 1$.

(10) Let $f\colon X \to Y$ be a function. For any subset $A \subseteq Y$, let $f^{-1}(A)$ be the pre-image of A in X. Show that this defines a function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Can you say when it is injective or surjective?

(11) Suppose that $f\colon X \to Y$ and let $A \subseteq X$ and $B \subseteq Y$.
   (a) Prove or give a counterexample: $f^{-1}(f(A)) \subseteq A$
   (b) Prove or give a counterexample: $B \subseteq f(f^{-1}(B))$
   (c) Prove or give a counterexample: $f(A \cup f^{-1}(B)) = f(A) \cup B$.
   (d) Prove or give a counterexample: $f^{-1}(f(A) \cap B) = A \cap f^{-1}(B)$