MTH 301 SMT2 Solutions

Q1 Thm $\sqrt[3]{3}$ is irrational.

Proof (by contradiction) suppose $\sqrt[3]{3} = \frac{a}{b}$ in lowest terms. Then $3 = \frac{a^3}{b^3} \Rightarrow 3b^3 = a^3$ so $3 \mid a^3$. Claim: if $3 \mid a^3$ then $3 \mid a$. Proof (of claim) (contrapositive). suppose $3 \nmid a$, then $a = 3n+1$ or $3n+2$, so $a^3 = 27n^3 + 27n^2 + 9n + 1$ or $a^3 = 27n^3 + 54n^2 + 36n + 8$, in either case $3 \nmid a^3$, as required $\boxed{\text{claim}}$ So $a = 3c$ for some $c \in \mathbb{Z}$, so $3b^3 = (3c)^3 = 27c^3$, so $b^3 = 9c^3$, so $9 \mid b^3$, so $3 \mid b^3$, and a, b have common factor # $\square$.

Q2 a) $f: \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ $(a,b) \mapsto 2^a 3^b$ injective but not surjective.

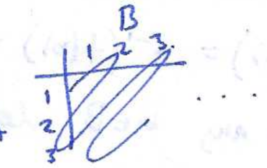
b) $(a,b) \mapsto 0$ neither injective nor surjective.

Q3 a) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = (x-1)x(x+1)$ surjective but not injective.

b) $f(x) = 0$ neither injective nor surjective.

Q4 a) $x \mapsto \frac{1}{x}$ b) $x \mapsto x - 1$ c) $x \mapsto \ln(x)$ d) $x \mapsto \ln(\frac{1}{x} - 1)$.

Q5 Claim if A and B are countable then $A \times B$ is countable. Proof:



$x^2 + ax + b$ has at most two solutions, so solutions indexed by $(a, b, \pm 1) \in \mathbb{Z} \times \mathbb{Z} \times \{\pm 1\}$ countable.

Q6 a) x is not a multiple of 3. $(\exists n \in \mathbb{N})(x = 3n)$

b) π is rational. $(\forall a, b \in \mathbb{N})(\pi \neq a/b)$.

c) f is injective $(\exists a, b \in A)(f(a) = f(b)$ and $a \neq b)$.

d) $n^2 + n$ is even. $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(n^2 + n \neq 2m)$

e) all subsets of $\mathbb{R}$ have an upper bound. $(\exists A \subset \mathbb{R})(\forall x \in \mathbb{R})(\exists a \in A)(x \leq a)$.

f) the set of primes is not bounded above. $(\exists n \in \mathbb{N})(\forall p \in \mathbb{N})(p \leq n$ or $(\exists q \in \mathbb{N})(q \mid p$ and $q \neq 1$ and $q \neq p))$

Q7 a) $(\exists a, b \in \mathbb{N})(\sqrt{\pi} = a/b)$.

b) $f: A \to B$. $(\exists a, b \in A)(f(a) = f(b)$ and $a \neq b)$ or $(\forall b \in B)(\exists a \in A)(f(a) = b)$.

c) Let $P = \{n \in \mathbb{N} \mid (\forall p)$ if $p \mid n$ then $p = 1$ or $p = n\}$. Let $D(n) = \{d \in \mathbb{N} \mid d \mid n\}$. $|D(n) \cap P| \neq 2$.

↑ exactly two.

$|D(n) \cap P| = 1$

at least 2

d) $(\exists f: A \to B)((\exists a, b \in A)\ f(a) = f(b)$ and $a \neq b)$.

e) $f: \mathbb{R} \to \mathbb{R}$ $(\exists a, b \in \mathbb{R})(a < b$ and $f(a) \geq f(b))$.

f) $\lim_{x\to 2} f(x) = 1$ $(\forall \epsilon>0)(\exists\delta>0)(|x-2|<\delta \Rightarrow |f(x)-1|<\epsilon)$.

negation: $(\exists\epsilon>0(\forall\delta>0)(|x-2|<\delta$ and $|f(x)-1|>\epsilon)$.

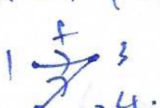
Q8 a) vacuously true. b) $S\neq\emptyset \Rightarrow P(S)\neq\emptyset$ true. c) $S=\emptyset \Rightarrow P(S)=\emptyset$ false. d) $P(S)\neq\emptyset \Rightarrow S\neq\emptyset$ false. e) true, not equivalent to ones above

Q9 a) Thm $\forall x\in\mathbb{Z}$ x^2+x is even. Proof $x^2+x = x(x+1)$, either x or $x+1$ is even, and a product of any number with an even number is even □.

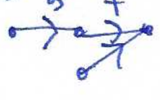
b) ~~Thm sum of any 4 consecutive integers is divisible by 4. Proof consider $x+(x+1)+(x+2)+(x+3)$ $= 4x+6$~~ false $x=0$, $0+1+2+3=6$ $4\nmid 6$.

c) Thm $f:A\to B$ has an inverse iff both surjective and injective. Proof (⇒) suppose $f^{-1}:B\to A$ with $f(f^{-1}(b))=b$ and $f^{-1}(f(a))=a$. claim f injective. suppose $f(a)=f(b)$, then $f^{-1}(f(a))=f^{-1}(f(b)) \Rightarrow a=b$. f surjective: given $b\in B$, $\exists f^{-1}(b)\in A$ s.t. $f(f^{-1}(b))=b$. □.

(⇐) for any $b\in B$ define $f^{-1}(b)$ to be ~~any~~ the unique element of $f^{-1}(\{b\})$. claim this defines the inverse function. check: as f surjective, for all $b\in B$, there is an $a\in A$ s.t. $f(a)=b$ so $f^{-1}(\{b\})\neq\emptyset$, so $f^{-1}(b)$ exists. check $|f^{-1}(\{b\})|=1$. suppose not, then $\exists a_1,a_2\in f^{-1}(\{b\})$ with $a_1\neq a_2$, but then $f(a_1)=f(a_2)=b$ # injective. So $f^{-1}(b)$ well defined. Finally, check $f(f^{-1}(b))=b$. Proof. $\{f^{-1}(b)\}=f^{-1}(\{b\})$ so $f(f^{-1}(b))=b$. check $f^{-1}(f(a)) = a$: $a\in f^{-1}(f(\{a\}))=f^{-1}(\{f(a)\})$ so $a=f^{-1}(f(a))$ as required. □.

d) false 

e) false  f injective $f\circ g$ not injective.

f) false  $f\circ g$ surjective, f surjective g not surjective.

g) ~~false $f\circ g$ increasing, f increasing, g not strictly increasing~~

Thm If $f\circ g$ strictly increasing and f strictly increasing, then g increasing

Proof (by contradiction) assume g not strictly increasing, then $\exists x<y$ with $g(x)\geq g(y)$. g strictly increasing ⇒ $f(g(x))\geq f(g(x))$ # $f\circ g$ strictly increasing. □.

h) false $x=0$ counterexample

Q10 $f: X \to Y$ defines a function $f^{-1}: P(Y) \to P(X)$

Proof Given $A \subseteq Y$ define $A \mapsto f^{-1}(A)$, check: • $f^{-1}(A) \subseteq X$ so $f^{-1}(A) \in P(X)$ for all $A \in P(Y)$. • $f^{-1}(A)$ is uniquely defined given A, so $f^{-1}: P(Y) \to P(X)$ is a function $\square$.

f injective $\Leftrightarrow$ f^{-1} surjective. f surjective $\Leftrightarrow$ f^{-1} injective. $\leftarrow$ prove these!

Q11 $f: X \to Y$ $A \subseteq X$, $B \subseteq Y$.

a) counterexample: $\begin{tikzcd} 1 \arrow[dr, "f"] & \\ & 3 \\ 2 \arrow[ur] & \end{tikzcd}$ $A = \{1\}$

b) counterexample: $\begin{tikzcd} 1 \arrow[r, "f"] & 2 \\ & 3 \end{tikzcd}$ $B = \{3\}$

c) counterexample: $\begin{tikzcd} 1 \arrow[r, "f"] & 2 \\ & 3 \end{tikzcd}$ $A = \{1\}$, $B = \{3\}$.

d) counterexample $\begin{tikzcd} 1 \arrow[dr, "f"] & \\ & 3 \\ 2 \arrow[ur] & \end{tikzcd}$ $A = \{1\}$, $B = \{3\}$.