Math 301 Introduction to Proof Fall 24 Midterm 1

Name: Solutions

- **Start each question on a fresh sheet of paper. Staple together in numerical order at the end of the exam.**

(1) For each of the following statements, find two distinct elements in the truth set, and two distinct elements not in the truth set. (Indicate clearly which are which.)
 (a) $\frac{a}{b} \in \mathbb{Z}$, where the universe is $\mathbb{Z} \times \mathbb{Z}$.
 (b) $A \cup B = \mathbb{Z}$, where the universe is all subsets of $\mathbb{Z}$.

(2) Consider the statement:
 If x and y are real numbers with $x < y$ then $x^2 < y^2$.
 Which, if any, of the following substitutions give a counter example.
 (a) $x = 1, y = 2$ (b) $x = -2, y = 1$ (c) $x = 1, y = -2$ (d) $x = 2, y = 1$

(3) Write out a careful proof of the fact that the cube of any odd number is odd.

(4) Prove or disprove the following statement: If $A \cap B = A \cap C$ then $B = C$.

(5) Prove or disprove the following statement: If $a \mid b$ and $b \mid c$ then $a \mid c$.

(6) Prove or disprove the following statement: $A - (B \cup C) = (A - B) \cup (A - C)$.

(7) Prove or disprove the following statement: If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.

(8) State which of the following statements, are true, vacuously true, or false.
 (a) For integers a, b, c and d, if $a \mid b$ and $c \mid d$ then $ab \mid cd$.
 (b) If x is an integer with $x^2 = 2$ then x is positive.
 (c) If x is a real number with $x^2 = 2$ then x is positive.
 (d) If $\mathcal{P}(A) \cap \mathcal{P}(B) = \varnothing$ then $A \cap B = \varnothing$.

(9) Write out a careful proof of the fact that if $a \mid b$ then $a^2 \mid b^2$.

(10) Consider the following theorem and proof. Is the theorem correct? If it is not correct give a counterexample. Is the proof correct? If the proof is not correct, explain why it is not correct, and then give a correct proof.

Theorem. *For any sets A and B, if $A \times A = A \times B$ then $A = B$.*

Proof. Suppose $(a, a) \in A \times A$. Then as $A \times A = A \times B$, this implies that $(a, a) \in A \times B$, so $a \in B$, therefore $A = B$, as required. □

Midterm 1		Overall	

MT1 Solutions

Q1 a) $(1,1),(2,1)$ in truth set $(1,2)\ (1,3)$ not in truth set

b) $(\phi,\mathbb{Z}),(\mathbb{Z},\mathbb{Z})$ in truth set $(\phi,\phi),(\phi,\{1\})$ not in truth set.

Q2 b)

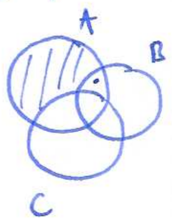
Q3 Thm If x is odd then x^3 is odd $(x\in\mathbb{Z})$.

Proof Suppose x is odd, then $x=2n+1$ for some integer n. Then $x^3=(2n+1)^3$ $=8n^3+12n^2+6n+1=2(4n^3+6n^2+3n)+1$ which is odd $\square$.

Q4 False. counterexample: $A=\phi$, $B=\phi$, $C=\{1\}$.

Q5 Thm If $a|b$ and $b|c$ then $a|c$.

Proof If $a|b$ then there is an integer m s.t. $b=ma$. If $b|c$ then there is an integer n such that $c=nb$. Then $c=nb=n.m.a$, as $n.m$ is an integer, c is a multiple of a, so $a|c$. $\square$.

Q6 

$A-(B\cup C)$

$(A-B)\cup(A-C)$

False. counterexample $A=B=\{1\}$, $C=\phi$.

$A-(B\cup C)=\phi$.

$(A-B)\cup(A-C)=\phi\cup\{1\}=\{1\}$.

Q7 Thm If $P(A)\subseteq P(B)$ then $A\subseteq B$.

Proof $A\subseteq A$ so $A\in P(A)$. As $P(A)\subseteq P(B)$, $A\in P(B)$, but this means $A\subseteq B$, as required $\square$.

Q8 a) F b) VT c) F d) VT

Q9 Thm If $a|b$ then $a^2|b^2$.

Proof If $a|b$ then there is an integer c such that $b=ac$. Then $b^2=a^2c^2$, and c^2 is an integer so $a^2|b^2$ $\square$.

Q10 False. counterexample, $A=\phi$, $B=\{1\}$.

Mistakes in proof include: (a,a) not arbitrary element of $A\times A$, doesn't consider $A\times A=\phi$, only does $A\times A\subseteq A\times B$ not both directions.