Mathematical Induction Fun Pack

1. Prove the following statements using either the First or Second Principle of Mathematical Induction. Be sure to state somewhere in your proof (probably best in the conclusion) which principle you used.
   (a) For all $n \in \mathbb{N}$, $1 + 3 + 5 + 7 + (2n - 1) = n^2$
   (b) For all $n \in \mathbb{N}$, $1 + 3 + 3^2 + ... + 3^{n-1} = \frac{3^n - 1}{2}$
   (c) For all $n \in \mathbb{N}$, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
   (d) For all $n \in \mathbb{N}$, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
   (e) For all $n \in \mathbb{N}$, $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \; + n \cdot 3^n = \frac{(2n-1)3^{n+1}+3}{4}$
   (f) For all $n \in \mathbb{N}$, $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$.
   (g) Prove that $n! > 3n$ for all $n \in \mathbb{N}_{[7,\infty)}$
   (h) Prove that $2n < n^2$ for all $n \in \mathbb{N}_{[4,\infty)}$
   (i) Show that given a set of $n + 1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.
   (j) Let F_k be the Fibonacci numbers defined by $F_1 = 1$, $F_2 = 1$, and for $k > 2$, $F_k = F_{k-1} + F_{k-2}$. Show that the following formulas hold for all $n \in \mathbb{N}$:
      (a) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$
      (b) $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$
      (c) $\sum_{k=1}^{n} F_k = F_{n+2} - 1$
      (d) $F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2}$.
   (k) Show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$ for all $n \in \mathbb{N}$.
   (l) Show that $2! + 4! + 6! + \cdots + (2n)! \leq ((n + 1)!)^n$ for all $n \in \mathbb{N}$.
   (m) Show that for all $n \in \mathbb{N}$, $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$, where there are n 2's in the expression on the left. [Hint: $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$]
   (n) Any convex polygon with n sides can be cut into $n - 2$ triangles. Prove it.
   (o) The triangle inequality states that the length of a triangles side is always smaller than the sum of the lengths of the other two sides of that triangle. Prove that a side length of
      (a) a quadrilateral
      (b) a pentagon
      (c) any n-sided polygon

      is less than the sum of all its other side lengths.
   (p) Let $x \in \mathbb{R}$. Suppose $x + \frac{1}{x}$ is an integer. Prove that $x^2 + \frac{1}{x^2}$ is an integer. Now prove that $x^n + \frac{1}{x^n}$ is an integer for all $n \in \mathbb{N}_{[2,\infty)}$. [Note that x need not be an integer itself: for example, $x = \frac{5+\sqrt{21}}{2}$ works.]

(q) Prove that for all $n \in \mathbb{N}$,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}.$$

(r) Show that for all $n \in \mathbb{N}$,

$$\sum_{\{a_1,a_2,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 \cdot a_2 \cdot a_3 \cdot \cdots \cdot a_n} = n.$$

For example, for $n = 2$, the sum is $\frac{1}{1} + \frac{1}{2} + \frac{1}{1 \cdot 2}$.

(s) Prove that $3^n < n!$ for all $n \in \mathbb{N}_{[6,\infty)}$.

(t) Consider the following four equations:

$$\begin{aligned} 1 &= 1 && (1) \\ 2+3+4 &= 1+8 && (2) \\ 5+6+7+8+9 &= 8+27 && (3) \\ 10+11+12+13+14+15+16 &= 27+64 && (4) \end{aligned}$$

Conjecture the general formula suggested by these four equations, and prove your conjecture.

(u) Prove that the product of two consecutive natural numbers is an even number.

(v) Prove that $n(n+1)(n+2)$ is divisible by 6 for all $n \in \mathbb{N}$.

(w) Prove that $1 + 4 + 7 + ... + (3n-2) = \frac{1}{2}n(3n-1)$ for all $n \in \mathbb{N}$.

(x) Prove that $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9 for all $n \in \mathbb{N}$.

(y) Prove that $(2n+7) < (n+3)^2$ for all $n \in \mathbb{N}$.

(z) Prove that $1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$ for all $n \in \mathbb{N}$.

(aa) Prove that $10^{2n-1} + 1$ is divisible by 11 for all $n \in \mathbb{N}$.

(ab) Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all $n \in \mathbb{N}$.

(ac) Prove that $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \in \mathbb{N}$.

(ad) Prove that the number of all the subsets of a set containing n distinct elements is 2^n for all $n \in \mathbb{N}$.

(ae) Prove that $7 + 77 + 777 + ... + 777...7(\text{n digits}) \;= \frac{7}{81}(10^{n+1} - 9n - 10$ for all $n \in \mathbb{N}$.

2. Now work problems (a) through (ae) above using the Well Ordering Principle.