recall: chain rule: $\frac{d}{dx}(f(g(x)) = f'(g(x)).g'(x)$

substitution / change of variable

$\int f(x)dx$, $x(u)$ $\quad \int_{x=a}^{x=b} f(x)dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u)) \frac{dx}{du} du$

3 things to change: function, differential, limits.

mnemonic: cancelling fractions $dx = \frac{dx}{du} du$

Q: why does this work?

$\int f(x)dx$, $x(u)$ $\quad$ set $F(x) = \int f(x)dx$, so $F'(x) = f(x)$

$\int f(x)dx = F(x)$ differentiate wrt u: $\frac{d}{du}(F(x(u))) = F'(x(u)).x'(u)$

$\int f(x)dx =$ integrate wrt u $= \int f(x(u))\, x'(u)\, du = \int f(x(u)) \frac{dx}{du} du.$

useful fact: $\frac{du}{dx} = 1 / \frac{dx}{du}$

Examples ① $\int_0^1 e^{-7x} dx$ $\quad$ set $u = -7x$

$\frac{du}{dx} = -7$

$\int_0^{-7} e^u \frac{dx}{du} du = \int_0^{-7} e^u . \frac{1}{-7} du = -\frac{1}{7} \int_0^{-7} e^u du = -\frac{1}{7}\left[e^u\right]_0^{-7} = -\frac{1}{7}(e^{-7} - 1)$

② $\int_0^2 x^2 \sqrt{x^3+1}\, dx$, $\int_0^{\pi/4} \tan^3 x \sec^2 x\, dx$ $\quad \int_0^1 \frac{x}{x^2+1} dx$

§5.8 More integrals

recall: $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ so $\int \frac{1}{x} dx = \ln|x| + c$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ $\quad \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$ $\quad\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + c$

$\frac{d}{dx}(e^x) = e^x$ $\quad\int e^x\,dx = e^x + c$

<u>recall</u> $b^x = e^{x\ln(b)}$ $\quad\int b^x\,dx = \int e^{x\ln(b)}\,dx = \frac{1}{\ln(b)} e^{x\ln(b)} + c$

(sub $u = x\ln(b)$).

<u>Examples</u>

① $\int \frac{1}{9+x^2}\,dx = \frac{1}{9}\int \frac{1}{1+x^2/9}\,dx = \frac{1}{9}\int \frac{1}{1+(x/3)^2}\,dx$ $\quad$ sub $u = x/3$ ...

② $\int \frac{1}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{1-(2x)^2}}$ $\quad$ sub $u = 2x$ ....

③ $\int x(x+1)^9\,dx$ ④ $\int \sqrt{4x-1}\,dx$ ⑤ $\int \sin^3 x \cos x\,dx$

⑥ $\int x\cos(x^2)\,dx$ ⑦ $\int x\sqrt{x^2-1}\,dx$ ⑧ $\int \frac{1}{x\ln(x)}\,dx$

⑨ $\int \frac{e^{2x}+e^{4x}}{e^x}\,dx$ ⑩ $\int 2^x\,dx$ ⑪ $\int x e^{x^2}\,dx$.