comparisons: $f(x) \leq g(x)$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

§5.3 Antiderivaties

Def^n A function $F(x)$ is an antiderivative for $f(x)$ if $F'(x) = f(x)$.

Example $f(x) = x^2$, $F(x) = \frac{1}{3}x^3$. note: $\frac{1}{3}x^3 + 1$ also works.

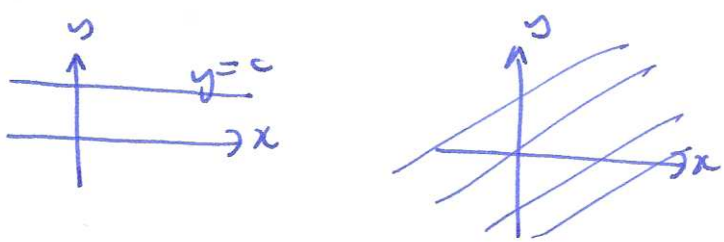
General antiderivative

Th^m Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x) + c$ for some $c \in \mathbb{R}$.

Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, then $F'(x) = G'(x) = f(x)$, so $(F(x) - G(x))' = f(x) - f(x) = 0$, so $F(x) - G(x) = c$, constant. $\square$.

Picture $f(x)$ gives the slope function for $F(x)$.

Example $f(x) = c$



$f'(x) = c$

$F(x) = cx + d$

Notation: $\int f(x)\,dx$ means general antiderivative $F(x) + c$

~~observation~~: **Example** $\int x^2 + \frac{1}{x} + \sin x \, dx = \frac{1}{3}x^3 + \ln|x| - \cos x + c$

Observation: every rule for differentiation gives a rule for integration.

warning: no easy analog of product, quotient, chain rule!

Alternate view: we can think of finding the indefinite integral as finding a function given its slope, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general, there is a family of solutions $F(x) + c$, but if we know the value of the solution we want at $x = 0$ (sometimes called an initial condition) then this picks out a particular solution.

<u>Example</u> motion under gravity, acceleration: $a(t) = x''(t) = -g$ (constant).

velocity: $v(t) = x'(t) = -gt + c$

if $v(0) = v_0$, initial velocity, then $v(t) = -gt + v_0$

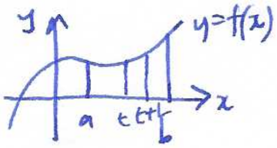
position: $x(t) = -\frac{1}{2}gt^2 + v_0 t + c$

if $x(0) = x_0$, initial position, then $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

§5.4 Fundamental theorem of calculus I

<u>Thm</u> (FTC ①) Suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an anti-derivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

<u>intuition</u>:



consider $\int_a^t f(x)\,dx$

Q: what is the rate of change wrt t?

$$\frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(x)\,dx$$

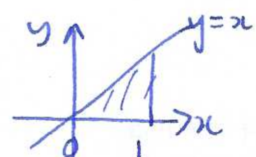
$$\approx \frac{\text{area of rectangle}}{h} = \frac{f(t)\times h}{h} = f(t)$$

i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$, so $\int_a^t f(x)\,dx = F(t) + c$

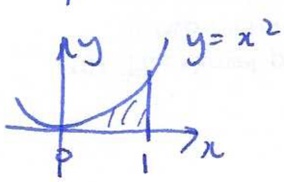
Q: what is the constant? $t = a$: $\int_a^a f(x)\,dx = 0 = F(a) + c \Rightarrow c = -F(a)$

so $\int_a^t f(x)\,dx = F(t) - F(a)$ $\square$.

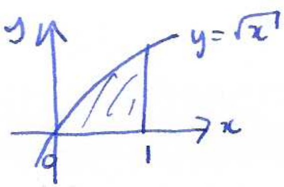
<u>Examples</u>



$$\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$



$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$



$$\int_0^1 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$