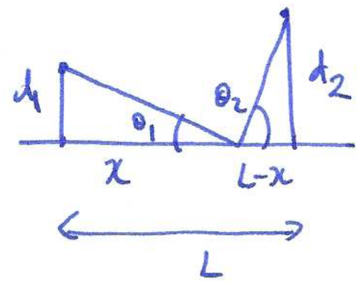
Example a ball bounces off a wall. show that the path which minimises distance has equal angles of incidence and reflection.

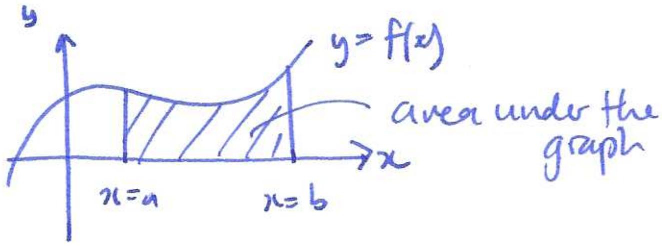


$$f(x) = \text{distance} = \sqrt{d_1^2+x^2} + \sqrt{d_2^2+(L-x)^2}$$

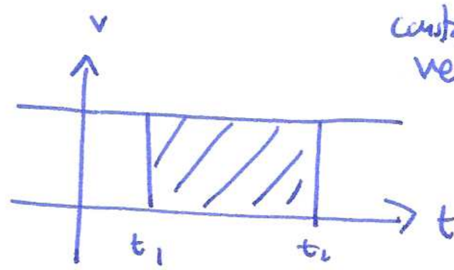
$$f'(x) = \tfrac{1}{2}(d_1^2+x^2)^{-1/2}.2x + \tfrac{1}{2}(d_2^2+(L-x)^2)^{-1/2}.2(L-x).(-1)$$

$f'(x)=0$:
$$\frac{x}{\sqrt{d_1^2+x^2}} = \frac{L-x}{\sqrt{d_2^2+(L-x)^2}} \iff \cos\theta_1 = \cos\theta_2$$
$$\Rightarrow \theta_1 = \theta_2.$$

§5.1 Approximating area



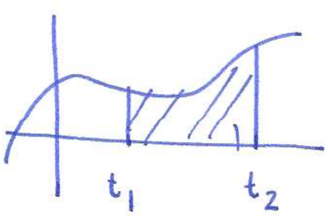
Example



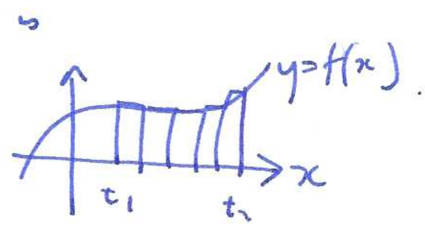
constant velocity

distance travelled = velocity × time
= area under the graph

non-constant velocity:



fact: distance travelled = area under the graph



finding the area: approximate by rectangles

Example

find area of graph under $y=x$ between $x=0$ and $x=1$
(answer $=\frac{1}{2}$)

approximate with 4 rectangles: area $\approx$ sum of $\underbrace{\text{area of rectangles}}_{\text{width}\times\text{height}}$

$$= \tfrac{1}{4}f(0) + \tfrac{1}{4}f(\tfrac{1}{4}) + \tfrac{1}{4}f(\tfrac{2}{4}) + \tfrac{1}{4}f(\tfrac{3}{4})$$
$$= \tfrac{1}{4}\left(f(0)+f(\tfrac{1}{4})+f(\tfrac{1}{2})+f(\tfrac{3}{4})\right) = \tfrac{1}{4}\sum_{i=0}^{3} f(\tfrac{i}{4})$$
$$= \tfrac{1}{4}\left(0+\tfrac{1}{4}+\tfrac{2}{4}+\tfrac{3}{4}\right) = \tfrac{1}{4}\cdot\tfrac{6}{4} = \tfrac{3}{8} \approx 0.375$$

approximate with n rectangles:

$$f(0)\tfrac{1}{n} + f(\tfrac{1}{n})\tfrac{1}{n} + f(\tfrac{2}{n})\tfrac{1}{n} + \cdots + f(\tfrac{n-1}{n})\tfrac{1}{n} = \sum_{i=0}^{n-1} \frac{1}{n} f(\tfrac{i}{n}) = \sum_{i=0}^{n-1} \frac{1}{n}\cdot\frac{i}{n}$$

$$= \frac{1}{n}\left(f(0) + f(\tfrac{1}{n}) + \cdots + f(\tfrac{n-1}{n})\right)$$

$$= \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right) = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

$$= \frac{1}{n^2}\left(0 + 1 + 2 + \cdots + n-1\right)$$

claim: $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$

proof ⓪ pairing: $1 + 2 + \cdots + n-1 + n$ $n+1$ $n+1$

n even: $\frac{1}{2}n(n+1)$

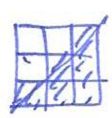
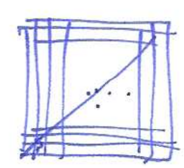
n odd: $\frac{n-1}{2}(n+1) + \frac{n+1}{2} = \frac{1}{2}n(n+1)$

① induction: assume true for some k: $s_k = 1+2+\cdots+k = \frac{1}{2}k(k+1)$

$$s_{k+1} \quad \underbrace{1+2+\cdots+k}_{s_k} + (k+1) = \tfrac{1}{2}k(k+1) + (k+1) = (k+1)(\tfrac{1}{2}k+1) = \tfrac{1}{2}(k+1)(k+2) \checkmark$$

so true for $k \Rightarrow$ true for $k+1$

check base case: $k=1$ $\quad s_1 = \frac{1}{2}\cdot 1.2 = 1$ ✓ □.

②   $\frac{1}{2}(2^2) + \frac{1}{2}.2$  $\frac{1}{2}(3^2) + \frac{1}{2}3$  $\frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1)$

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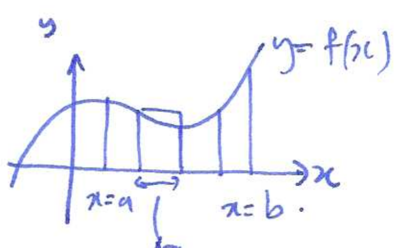
so approximate area is $\frac{1}{n^2}\left(1+2+\cdots+(n-1)\right) = \frac{1}{n^2}\frac{1}{2}(n-1).n = \frac{1}{2}\frac{n^2-n}{n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right)$

$\rightarrow \frac{1}{2}$ as $n \to \infty$.

Notation



if N rectangles of equal width then $\Delta x = \frac{b-a}{N}$

left endpoint rectangles $\quad L_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$

right endpoint rectangles $\quad R_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$