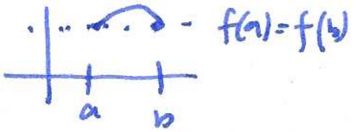
Thm (Rolle's Thm) suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$ then there is a $c\in(a,b)$ s.t. $f'(c)=0$

Proof

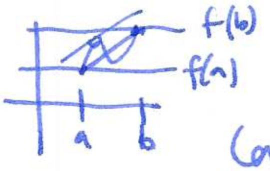


- if there is a local max/min at c then $f'(c)=0$
- if no local max/min, then $f(x)=\text{const}=f(a)=f(b)$

so $f'(c)=0$ for all $c\in(a,b)$ □.

§4.3 First derivative test

Thm (MVT) (mean value theorem) suppose f is cts on $[a,b]$ and differentiable on (a,b), then there is $c\in(a,b)$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$, i.e. there is a point where the slope is equal to the average rate of change.



Proof (Rolle's theorem turned sideways) □.

Corollary If $f(x)$ is differentiable and $f'(x)=0$ then $f(x)=c$ constant.

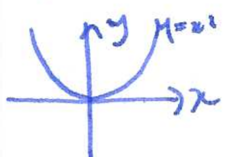
Proof suppose there is $a\neq b$ with $f(a)\neq f(b)$, then there is $c\in(a,b)$ with $f'(c)=\dfrac{f(b)-f(a)}{b-a}\neq 0$. □.

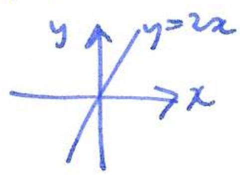
Monotonicity suppose f is differentiable on (a,b):

If $f'(x)>0$ for all $x\in(a,b)$ then f is increasing on (a,b).

If $f'(x)<0$ for all $x\in(a,b)$ then f is decreasing on (a,b)

Example $f(x)=x^2$ $\quad f'(x)=2x$





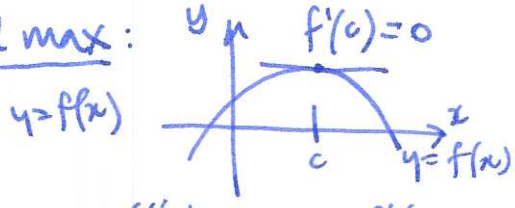
f increasing on $(0,\infty)$
decreasing on $(-\infty,0)$.

② $f(x)=x^2-2x-3$
$f'(x)=2x-2$

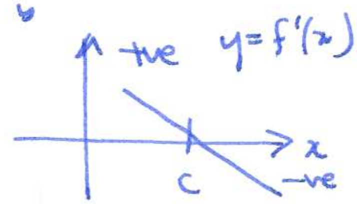
$f'(x)$ where $2x-2>0$
$x>1$

First derivative test

local max:

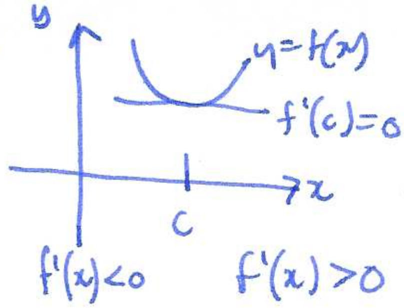


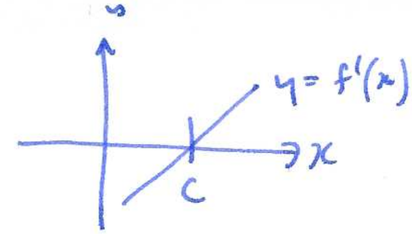
$f'(x)>0$ $\qquad$ $f'(x)<0$



if $f'(x)$ goes from positive to negative at c ⇒ c is local max.

local min :





if $f'(x)$ goes from negative to positive $\Rightarrow$ local min.

Thm **First derivative test** If $f(x)$ is differentiable and $f'(c) = 0$

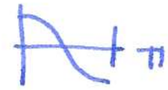
then if $f'(x)$ changes from +ve to −ve at c $\Rightarrow$ c local max

−ve to +ve $\Rightarrow$ c local min.

Example classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0, \pi]$

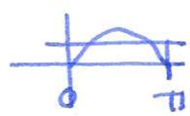
· find critical points : $f'(x) = -2\cos(x)\sin(x) + \cos x$

· solve $f'(x) = 0$: $\cos(x)(1 - 2\sin x) = 0$ · $\cos(x) = 0$



$x = \frac{\pi}{2}$

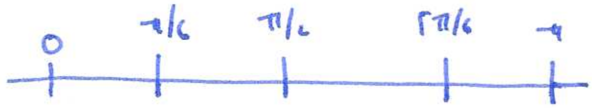
· $1 - 2\sin(x) = 0 \Leftrightarrow \sin x = \frac{1}{2}$



$x = \frac{\pi}{6}, \frac{5\pi}{6}$

so critical points are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

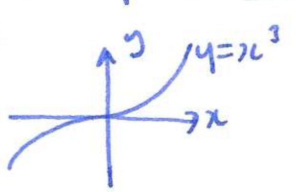
find sign of $f'(x)$:



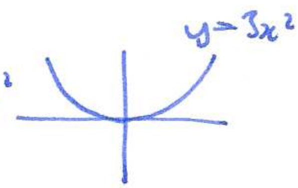
$\cos(x)$	+	+	−	−
$1 - 2\sin(x)$	+	−	−	+
$f'(x)$	+	−	+	−
	local max	local min	local max	

Example critical point <u>not</u> max or min

$f(x) = x^3$

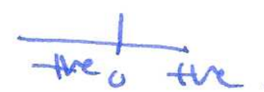


$f'(x) = 3x^2$



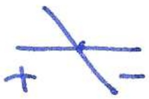
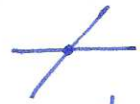
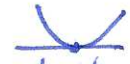
$f'(x) = 0 \Rightarrow x = 0$

$f'(x)$



$\Rightarrow x = 0$ <u>not</u> local max or min.

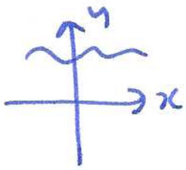
<u>recall</u> critical point $f'(x) = 0$ $f(x)$

$f'(x)$  $+$ $-$  $-$ $+$  $+$ $+$ or  $-$ $-$

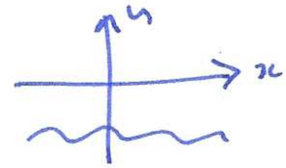
$f''(x)$ $-$ve $+$ve 0

§4.4 Second derivative test

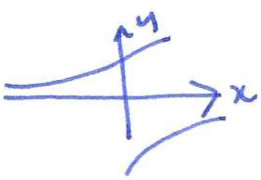
recall $f(x) > 0$
f positive



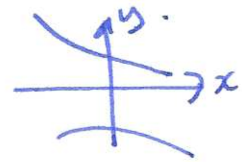
$f(x) < 0$
f negative



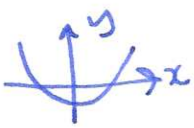
$f'(x) > 0$
f increasing



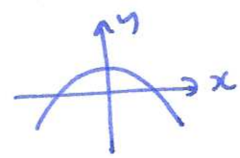
$f'(x) < 0$
f decreasing



$f''(x) > 0$
concave up

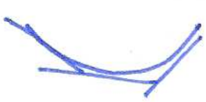


$f'(x) < 0$
concave down



mnemonic ${\ddot{\smile}}$ up ${\ddot{\frown}}$ down

Q: how do the slopes change?



concave up
$\leftrightarrow$ slopes increasing
$\leftrightarrow$ $f''(x) > 0$

concave down
$\leftrightarrow$ slopes decreasing
$\leftrightarrow$ $f''(x) < 0$

Defn Let $f(x)$ be differentiable on an interval (a,b) then

$f(x)$ is concave up $\Leftrightarrow$ $f''(x) > 0$

$f(x)$ is concave down $\Leftrightarrow$ $f''(x) < 0$

Defn An <u>inflection point</u> is where the graph changes from concave up to concave down, or vice versa.

Note inflection point $\Rightarrow$ $f''(x) = 0$
$\not\Leftarrow$ $f''(x) = 0$.