Examples ① $e^{4x} = f(g(x))$ where $f(x) = e^x \quad f'(x) = e^x$
$g(x) = 4x \quad g'(x) = 4$

so $(e^{4x})' = f'(g(x)) \cdot g'(x) = e^{4x} \cdot 4$

② $\sin^2(x) = f(g(x))$ where $f(x) = x^2 \quad g(x) = \sin x$
$f'(x) = 2x \quad g'(x) = \cos x$

so $(\sin^2(x))' = 2\sin x \cos x$.

③ $\sqrt{x^3+1}$, etc.

Alternate notation $\quad f(g(x)) \Leftrightarrow f(u), \quad u = g(x)$

$$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$$ mnemonic: "cancelling fractions"

Examples $\quad \cos(x^2), \; e^{\sqrt{x}}, \; \sin\left(\frac{\pi x}{180}\right), \; \sqrt{x+\sqrt{x^2+1}}$

Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$ [answer should be $f'(g(x)) \cdot g'(x)$]

write this as $$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \; \frac{g(x+h) - g(x)}{h}$$

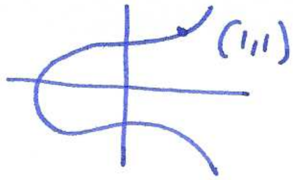
set $k = g(x+h) - g(x)$, as g is cts, $h \to 0 \Rightarrow k \to 0$

so $$= \lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} \; \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(g(x)) \cdot g'(x) \; \square$$

Examples
$$\frac{d}{dx}\left(g(x)^n\right) = n(g(x))^{n-1} \cdot g'(x).$$
$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x)$$
$$\frac{d}{dx}\left(f(ax+b)\right) = a f'(ax+b)$$

§3.8 Implicit differentiation

suppose $y^4 + xy = x^3 - x + 2$



can't solve for y explicitly

but can think of y as function of x, $y(x)$.

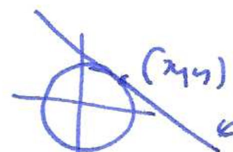
$$(y(x))^4 + x \cdot y(x) = x^3 - x + 2 \quad \leftarrow \text{differentiate wrt } x \text{ using chain rule.}$$

$$4(y(x))^3 \cdot y'(x) + y(x) + xy'(x) = 3x^2 - 1$$

$$y'(x)\left(4y^3 + x\right) = 3x^2 - 1 - y \qquad y'(x) = \frac{3x^2 - 1 - y}{4y^3 + x}$$

<u>Example</u> $x^2 + y^2 = 1$

$2x + 2yy' = 0 \qquad y' = -x/y$



<u>Application</u>: derivatives of inverse functions

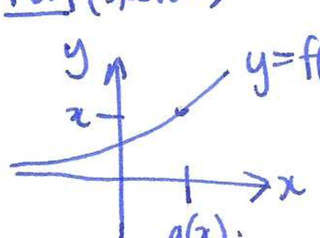
<u>Example</u> $y = \ln(x)$

$e^y = x$

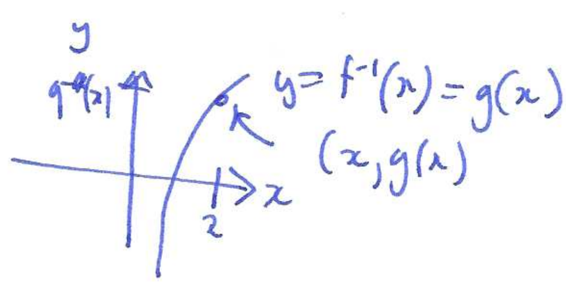
$e^y \cdot y' = 1 \qquad y' = e^{-y} = e^{-\ln(x)} = \frac{1}{x}$

<u>Thm</u> If $f(x)$ is differentiable and one-to-one, with inverse $f^{-1}(x) = g(x)$, then $g'(x) = \frac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

<u>Proof</u> (sketch)



reflect in $y=x$ $\longrightarrow$



$\frac{1}{\text{slope}}$ at $g(x)$, i.e. $\frac{1}{f'(g(x))}$ $\longleftarrow$ slope at x: $g'(x)$ $\square$.

<u>Application</u>: derivatives of inverse trig functions