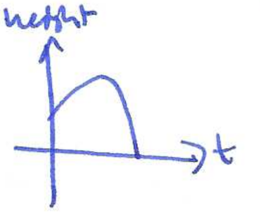
Motion under gravity



$s_0 = s(0) =$ height at $t=0$

$v_0 = v(0) =$ velocity at $t=0$

$s''(t) = v'(t) = a(t) = -g$ $\qquad g = 9.8 m/s^2$ (constant) $32 ft/s^2$

$s'(t) = v(t) = -gt + v_0$

$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$

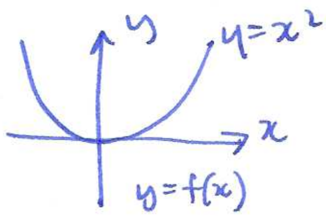
Example throw a stone upwards at $10m/s$ from height $2m$, what is max height?

$s(t) = 2 + 10t - \frac{1}{2}gt^2$

$v(t) = 10 - gt \qquad v(t) = 0 \Rightarrow t = \frac{10}{g} \simeq 1 \qquad s(1) = 2+10-5 = 7m.$

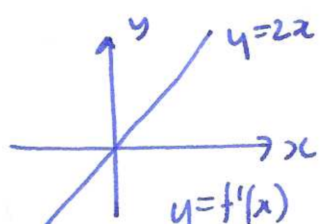
- ~~when does it hit the ground~~?
- how fast is it going when it hits the ground?
- if I can throw a stone 10m high, how fast can I throw it?

§3.5 Higher derivatives



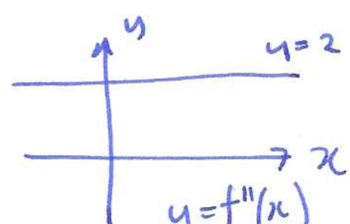
$y=f(x)$

"distance"



$y=f'(x)$

slope (rate of change of $f(x)$)
"velocity"



$y=f''(x)$

slope of $f'(x)$
"acceleration".

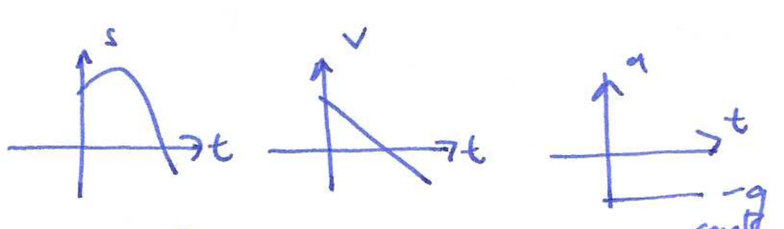
Example $f(x) = xe^x$

$f'(x) = xe^x + e^x$

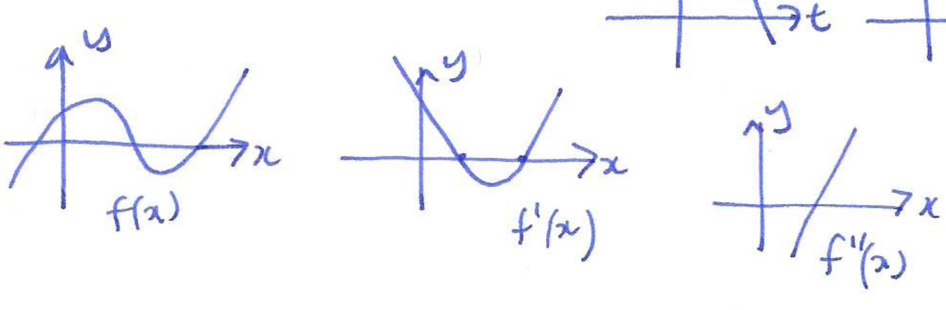
$f''(x) = xe^x + e^x + e^x = xe^x + 2e^x$

$f^{(3)}(x) = f'''(x) = xe^x + e^x + 2e^x = xe^x + 3e^x.$ etc.

Example acceleration due to gravity

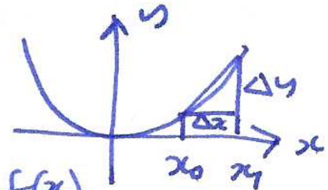


Example



§3.4 Rates of change

recall: average rate of change $= \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$



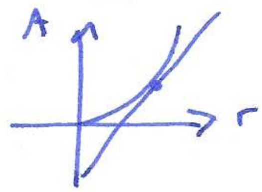
instantaneous rate of change: $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

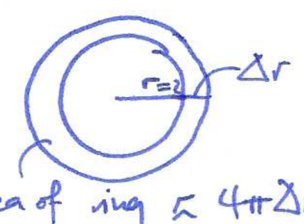
<u>observation</u>: if Δx is small, we can use the average rate of change to approximate the actual rate of change, and vice versa.

<u>Example</u> area of circle is πr^2 $\quad A = \pi r^2$

calculate rate of change of area wrt radius $\dfrac{dA}{dr} = 2\pi r$

e.g. $\left.\dfrac{dA}{dr}\right|_{r=2} = 4\pi \qquad \left.\dfrac{dA}{dr}\right|_{r=5} = 10\pi$





area of ring $\approx 4\pi\Delta r$

for small $\Delta r, h$: $\quad f'(x_0) \approx \dfrac{f(x_0+h)-f(x_0)}{h}$

so $f(x_0+h) \approx f(x_0) + hf'(x_0)$. ← linear approximation formula.

<u>Example</u> stopping distance in feet given by $F(s) = 1.1s + 0.05s^2$

(s speed in mph) calculate stopping distance when $s=30$, $F(30)=78$

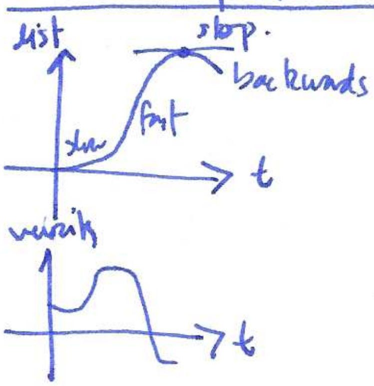
$F'(s) = 1.1 + 0.1s \qquad F'(30) = 1.1 + 0.1\times 30 = 4.1$ ft/mph.

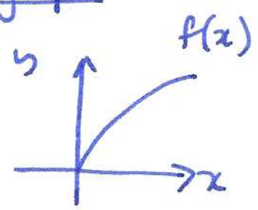
estimate stopping distance at $s=31$ (using above info)

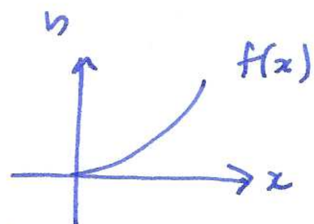
$F(s+h) \approx F(s) + hF'(s)$

$F(31) \approx F(30) + 1.F'(30) = 78 + 4.1 = 82.1$

<u>On the interpretation of graphs</u>



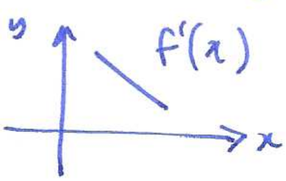


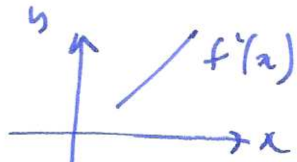


← increasing functions

rate of change

decreasing $\qquad$ increasing





§3.6 Trigonometric functions

Thm $\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$

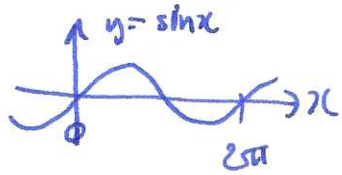
recall: $\sin(A+B) = \sin A \cos B + \cos A \sin B$

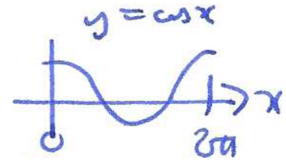
Proof (for $\sin x$) $\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0} \sin x \cdot \frac{\cos(h)-1}{h} + \lim_{h\to 0} \cos x \frac{\sin h}{h}$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{1} = \cos x \quad \square.$$

Q: can this be right?





Example $f(x) = x \sin x$

$f'(x) = x\cos x + \sin x$

Thm $\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Proof (of $\tan x$) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$

$= \sec^2 x \quad \square.$

Example $\frac{d}{dx}(e^x \cos x) = e^x \cdot -\sin x + e^x \cos x$

§3.7 Chain rule

composition of functions : $f(g(x)) = (f \circ g)(x)$

Examples e^{4x}, $\sin^2(x)$, etc. ...

Thm If f, g differentiable functions, then $f \circ g$ is differentiable, and

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

mnemonic : $(f(g(x)))' =$ outside$'$(inside)(inside)$'$

note $f(g(x))$