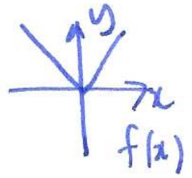
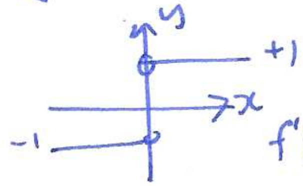
Thm Differentiable $\Rightarrow$ continuous (warning: continuous $\not\Rightarrow$ differentiable)

Example $f(x) = |x|$ cts





$f'(x)$ not cts at $x=0$

claim $f(x) = |x|$ not differentiable at $x=0$

check: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$

$\lim_{h\to 0+} \frac{|h|}{h} = +1 \quad \lim_{h\to 0-} \frac{|h|}{h} = -1 \Rightarrow \lim_{h\to 0} \frac{|h|}{h}$ DNE. D.

local picture if $f(x)$ is differentiable at $x=c$, means

$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists, (want to show: $\lim_{x\to c} f(x) = f(c)$)

consider $f(c+h)-f(c) = h \cdot \frac{f(c+h)-f(c)}{h}$ so $\lim_{h\to 0} f(c+h)-f(c)$

$= \lim_{h\to 0} h \cdot \lim \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} h \cdot f'(c) = 0 \cdot f'(c) = 0$. $\square$.

§3.3 Product and quotient rules

new functions from old: $f(x)g(x) \leftarrow$ product $\quad \frac{f(x)}{g(x)} \leftarrow$ quotient

Thm (product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)\,g + f\frac{d}{dx}(g)$$

warning: $(fg)' \neq f'g'$!!!

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x) \cdot x + x\frac{d}{dx}(x) = x + x = 2x$.

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = (3x^2)'(x^2+1) + (3x^2)(x^2+1)' = 6x^2(x^2+1) + 3x^2(2x)$

③ $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$.

Proof (of product rule) (assume f, g differentiable at x)

$$(fg)'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x)$$

$$= f(x)g'(x) + f'(x)g(x) \quad \square .$$

Thm Quotient rule (assume f, g differentiable at x, $g(x) \neq 0$)

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Examples ① $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - (x)(x+1)'}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

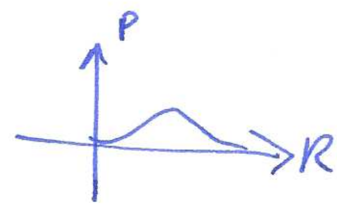
② $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)(e^t)' - (e^t+t)'e^t}{(e^t+t)^2} = \frac{(e^t+t)e^t - (e^t+1)e^t}{(e^t+t)^2} = \frac{te^t - e^t}{(e^t+t)^2}$

③ application: battery power

R resistance

r internal resistance

power $P = \frac{V^2R^2}{(r+R)^2}$



Q: when does the battery give maximal power?

A: when $\frac{dP}{dR} = 0$ $\quad P = \frac{V^2R^2}{(r+R)^2}$ $\quad P(R)$, V, r constant

$$\frac{dP}{dR} = \frac{(r+R)^2(V^2R)' - ((r+R)^2)'(V^2R)}{(r+R)^4} = \frac{(r+R)^2V^2 - (r^2+2Rr+R^2)V^2R}{(r+R)^4}$$

$$= \frac{V^2\left[(r+R)^2 - R(2r+2R)\right]}{(r+R)^4} = \frac{V^2(r+R)(r+R-2R)}{(r+R)^4} = \frac{V^2(r-R)}{(r+R)^3} = 0 \Rightarrow R = r$$

$\square$.