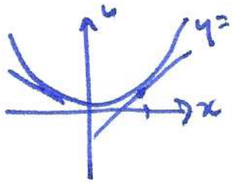
§3.2 Derivative as a function

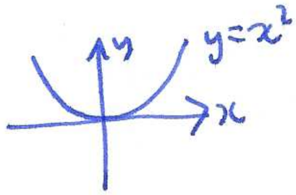


$f:\mathbb{R}\to\mathbb{R}$
$x \mapsto x^2$

at each point x, there is a tangent line, with a slope, which is a number.
Therefore we can define a function $\mathbb{R}\to\mathbb{R}$
$x \mapsto$ slope of tangent line at $(x, f(x))$

notation: we call this function $f'(x)$ "the derivative of f"

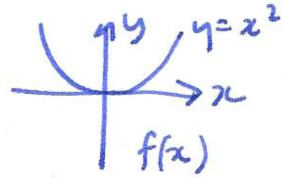
Example

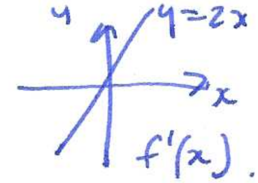


then $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ if $f(x)=x^2$

then $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h} = \lim\limits_{h\to 0} 2x+h = 2x$.

summary if $f(x) = x^2$, then $f'(x) = 2x$

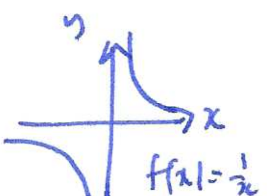


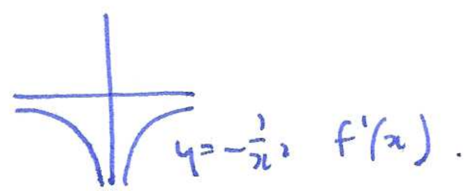


Example $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{x-(x+h)}{(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$$





Remarks

① functions f: domain $\to$ range, e.g. $f:\mathbb{R}\to\mathbb{R}$

derivative : (differentiable) functions $\longrightarrow$ functions

$f(x) \longmapsto f'(x)$

warning: not all functions differentiable.

② "calculus" means rules for doing calculations. We don't have to explicitly compute limits all the time.

Example $f(x) = x^3$ $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} \dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim\limits_{h\to 0} 3x^2+3xh+h^2 = 3x^2$.

Thm (powers of x) If $f(x) = x^n$ then $f'(x) = nx^{n-1}$ (works for all $n \in \mathbb{R}$)!

Examples $\frac{d}{dx}(x^2) = 2x$ $\frac{d}{dx}(x^3) = 3x^2$ $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$

$\frac{d}{dx}(x^1) = 1$ $\frac{d}{dx}(x^0) = 0$ $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

Thm $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof $f(x) = x^n$ $f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem: $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

all of these contain h^2.

so $\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + o(h^2) - x^n}{h}$

$= \lim_{h \to 0} nx^{n-1} + o(h) = nx^{n-1}$. □.

warning: this rule works for polynomials only, not exponentials.

$f(x) = x^{100}$ $f'(x) = 100x^{99}$ $f(x) = 2^x$ not polynomial

other useful rules

Thm (linearity) If f and g are differentiable functions, then:

- $f+g$ is differentiable with $(f+g)' = f' + g'$

$\leftrightarrow \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

- k constant $(kf)' = kf' \leftrightarrow \frac{d}{dx}(kf) = k\frac{df}{dx}$

Proof (follows from limit laws)

$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$

$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$= f'(x) + g'(x).$

$$(kf)'(x) = \lim_{h \to 0} \frac{(kf)(x+h) - (kf)(x)}{h} = \lim_{h \to 0} \frac{k f(x+h) - k f(x)}{h} = k \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

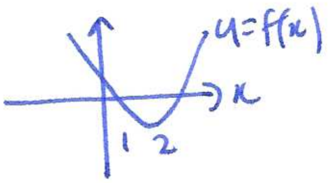
$= k f'(x)$ $\square$.

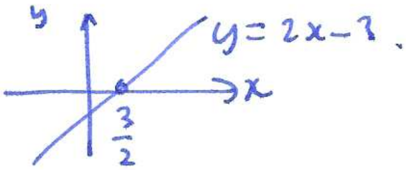
<u>Example</u> $f(x) = x^2 - 3x + 2$. Find $f'(x)$.

$$\frac{df}{dx} = \frac{d}{dx}(x^2 - 3x + 2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$$

$$= 2x - 3 + 0 = 2x - 3.$$

<u>graphs</u>





<u>Derivative of e^x</u> consider $f(x) = b^x$ $(b > 0)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} b^x \cdot \frac{b^h - 1}{h}$$

$$= b^x \underbrace{\lim_{h \to 0} \frac{b^h - 1}{h}}$$

doesn't depend on x!
assume this limit exists and call it m_b

we have shown: for exponential functions, the derivative is <u>proportional</u> to the value of the original function, i.e. $f(x) = b^x$ then $f'(x) = b^x \cdot m_b$
in particular, the slope at $x = 0$ is m_b.

<u>recall</u> e is defined to be the special number s.t. the slope of e^x at $x = 0$ is equal to 1, therefore $\boxed{\text{if } f(x) = e^x \text{ then } f'(x) = e^x}$ $\frac{d}{dx}(e^x) = e^x$.

<u>Example</u> $\frac{d}{dx}(7e^x + 8x^2) = 7e^x + 16x.$

<u>observation</u>: this shows that e^x is <u>not</u> a polynomial.

$\frac{d^n}{dx^n}(p(x)) \leftarrow$ degree goes down, eventually get zero.