$$\underbrace{\tfrac{1}{2}\sin\theta \le \tfrac{1}{2}\theta}_{\frac{\sin\theta}{\theta} \le 1} \le \tfrac{1}{2}\frac{\sin\theta}{\cos\theta} \qquad \text{so}$$

$$\underbrace{\tfrac{1}{2}\theta \le \tfrac{1}{2}\frac{\sin\theta}{\cos\theta}}_{\cos\theta \le \frac{\sin\theta}{\theta}}$$

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$

$$\lim_{\theta\to 0}\cos\theta = 1 \qquad \lim_{\theta\to 0} 1 = 1$$

squeeze thm $\Rightarrow$ $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ $\square$.

<u>Examples</u> ① $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{2x}$ know: $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$

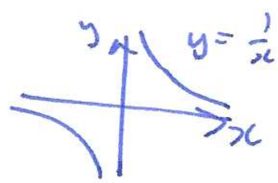
write $3x=\theta$ $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{2\cdot\theta/3} = \lim_{x\to\infty}\frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2}\lim\frac{\sin\theta}{\theta} = \frac{3}{2}$

② $\displaystyle\lim_{t\to 0}\frac{1-\cos t}{\sin t} = \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\frac{\cancel{\sin t}\,t}{\sin t} = \underbrace{\lim_{t\to 0}\frac{1-\cos t}{t}}_{0}\cdot\underbrace{\lim_{t\to 0}\frac{t}{\sin t}}_{1}$

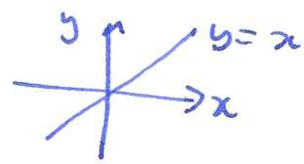
$= 0$.

<u>§2.7 Limits at infinity</u>

key observation: $\displaystyle\lim_{x\to\infty}\frac{1}{x} = 0$



$\displaystyle\lim_{x\to\infty} x = +\infty$



<u>Examples</u> ① $\displaystyle\lim_{x\to\infty}\frac{3x}{2x-1} = \lim_{x\to\infty}\frac{3}{2-1/x} = \frac{3}{2}$

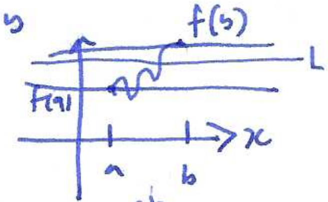
② $\displaystyle\lim_{x\to\infty}\frac{x^2+x}{x-3} = \lim_{x\to\infty}\frac{x+1}{1-3/x} = +\infty.$

③ $\displaystyle\lim_{x\to\infty}\frac{1}{x}-\frac{2}{3x+1} = \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{2}{3x+1} = 0-0=0$

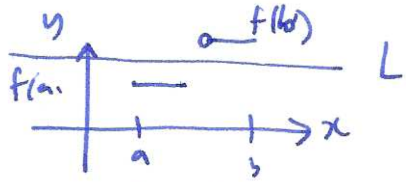
④ $\displaystyle\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{4x+1} = \lim_{x\to\infty}\frac{\sqrt{2+1/x^2}}{4+1/x} = \frac{\sqrt{2}}{4}$ (do $\displaystyle\lim_{x\to-\infty}$!)

§2.8 Intermediate Value Theorem (IVT)

"continuous functions can't skip values"



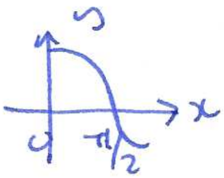
cts



not continuous

<u>Thm</u> (Intermediate value theorem IVT) If $f(x)$ is a cts function on a closed interval $[a,b]$ with $f(a) \neq f(b)$ then for any number L between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = L$.

<u>Example</u> show $\cos(x) = \frac{1}{4}$ has at least one solution



consider $[0, \frac{\pi}{2}]$ $\cos(0) = 1$ $\cos(\frac{\pi}{2}) = 0$ $0 \leq \frac{1}{4} \leq 1$

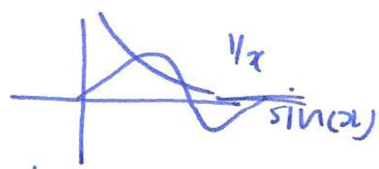
IVT $\Rightarrow \exists\, c \in [0, \frac{\pi}{2}]$ s.t. $f(c) = \frac{1}{4}$.

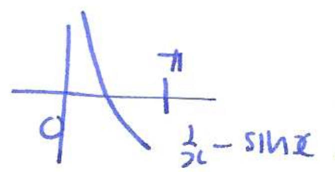
<u>special case: finding zeros</u>

<u>Corollary</u> if $f(x)$ is cts on $[a,b]$ and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ with $f(c) = 0$

<u>Bisection method</u>: find a solution to $\sin(x) = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin x$

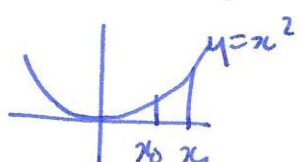




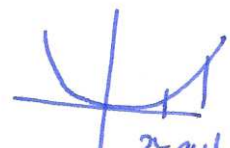
$f(0) = +\infty > 0$ $f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 < 0$, midpoint $\frac{\pi}{4}$, $f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.566 > 0$, so at ~~least~~ least one solution in $[\frac{\pi}{4}, \frac{\pi}{2}]$, etc.

§3.1 Defn of the derivative

<u>Recall</u>: we can compute the average rate of change of a function over an interval



$[x_0, x_1]$ $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$



$\frac{f(x+h) - f(x)}{h}$

<u>Q</u>: how do we find the slope of the tangent line?

<u>A</u>: look at the average rate of change over a small interval $[x, x+h]$ and take limit as $h \to 0$.

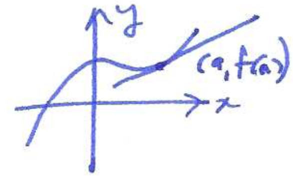
<u>Def^n</u> the slope of the tangent line at $x=a$ is $\lim\limits_{x\to 0} \frac{f(a+h)-f(a)}{h}$

<u>Notation</u> also called the <u>derivative at a</u>, written $f'(a)$ (Newton) or $\frac{df}{dx}(a)$ (Liebnitz)

If this limit exists, we say that the function $f(x)$ is differentiable at a.

<u>Note</u> $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$ same as $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$.

<u>Def^n</u> the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$. The equation for this line is:



$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x-a) \quad \text{or} \quad y = f(a) + f'(a)(x-a)$$

<u>Example</u> find the tangent line to $y = x^2$ at $x = 1$

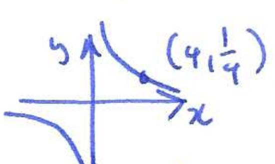
$(x, f(x)) = (1,1)$ slope $f'(1) = \lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \frac{(1+h)^2-1}{h}$

$= \lim\limits_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim\limits_{h\to 0} 2+h = 2$

so equation of tangent line is $y-1 = 2(x-1)$ $\quad y = 1+2(x-1)$.

or $y = 2x-1$

<u>Example</u> find the slope of the tangent line to $f(x) = \frac{1}{x}$ at $x=4$.



slope $f'(4) = \lim\limits_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$= \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{1}{4+h} - \frac{1}{4}\right) = \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{4-(4+h)}{4(4+h)}\right) = \lim\limits_{h\to 0} \frac{-h}{h\,4(4+h)}$

$= \lim\limits_{h\to 0} \frac{-1}{4(4+h)} = -\frac{1}{16}$ tangent line: $y - \frac{1}{4} = -\frac{1}{16}(x-4)$

<u>Example</u> find slope of tangent line $y = mx+b$. Find slope at $x=a$:

$f'(a) = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \frac{m(a+h)+b-(ma+b)}{h} = \lim\limits_{h\to 0} \frac{ma+mh+b-ma-b}{h}$

$= \lim\limits_{h\to 0} m = m$. <u>observation</u> if $f(x) = b$ (constant) then $f'(x) = 0$ for all x.