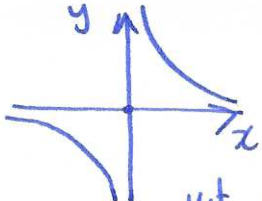
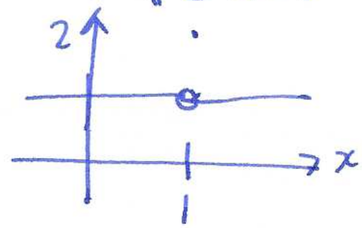
Examples



$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & \text{at } x = 0 \end{cases}$

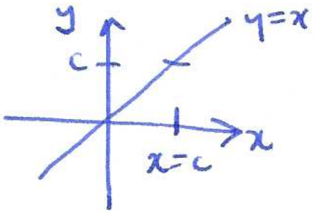
not cts at $x = 0$

$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1 \end{cases}$



not cts at $x = 1$

Example show $f(x) = x$ is cts.



$f(c) = c$, want to show $\lim_{x \to c} f(x) = c$

follows from limit laws: $\lim_{x \to c} x = c$ ✓.

Corollary polynomials / rational functions are cts where defined.

Defn $f(x)$ is <u>left continuous</u> at $x = c$ if $\lim_{x \to c-} f(x) = f(c)$

<u>right continuous</u> $\lim_{x \to c+} f(x) = f(c)$

If at least one of the right or left limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x = c$.

Building continuous functions

Thm 0 $f(x) = k$, $f(x) = x$ are continuous

Thm 1 suppose $f(x), g(x)$ both continuous at $x = c$, then the following functions are cts at $x = c$:

1) $f(x) + g(x)$ 2) $kf(x)$ for any constant k

3) $f(x)\, g(x)$ 4) $\frac{f(x)}{g(x)}$ if $g(c) \neq 0$

Proof: these follow immediately from the limit laws.

check 1) $f(x)$ cts means $\lim_{x \to c} f(x) = c$

$g(x)$ is cts means $\lim_{x \to c} f(x) = f(c)$

so $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$ as required □

__Thm 2__ Polynomials are continuous, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Rational functions $\frac{p(x)}{q(x)}$ are cts, except where $q(x) = 0$.

__Proof__ $f(x) = x$ is continuous. so $f(x).f(x) = x.x = x^2$ is cts. (products).

similarly, x^n is cts.

so $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is cts (multiplication by constant and addition)

so $\frac{p(x)}{q(x)}$ is cts (quotients) where $q(x) \neq 0$ □.

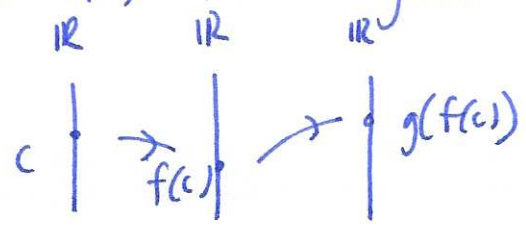
__useful facts__

__Thm 3__
- $\sin(x)$, $\cos(x)$ are continuous.
- b^x is cts.
- $\log_b(x)$ is cts
- $x^{1/n}$ is cts

(combinations of these with polynomials are sometimes called elementary functions)

__Thm 4__ (inverse functions) If $f: D \to \mathbb{R}$ is cts, with inverse $f^{-1}: D \to \mathbb{R}$ then f^{-1} is cts ($D \subseteq \mathbb{R}$)

__Thm 5__ (composition) If $f(x)$ is cts at $x = c$, and $g(x)$ is cts at $x = f(c)$ then $g(f(x))$ is cts at $x = c$



__Example__ $f(x) = \frac{2^x + \sin(x)}{\sqrt{x^2 + x + 1}}$ cts at $x = 1$

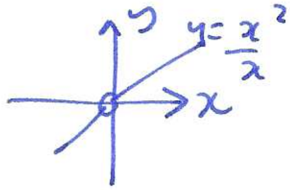
Q: where is $f(x)=\frac{x^2}{\sin(x)}$ cts?

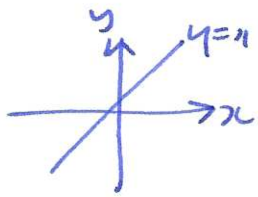
§2.5 Evaluating limits algebraically

Example $\frac{x^2}{x}$ undefined at $x=0$ $\frac{0}{0}$ ← indeterminate form

but $\lim_{x\to 0}\frac{x^2}{x}$ does not depend on value at $x=0$

$\frac{x^2}{x}=x$ for $x\neq 0$





so $\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0$

indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$, 0^0

note: $\frac{1}{0}$ not indeterminate, limit will be $\pm\infty$ or DNE

Examples ① $\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}$ $x=3$: $\frac{9-12+3}{9+3-12}=\frac{0}{0}$

factor: $\frac{(x-3)(x-1)}{(x-3)(x+4)}=\frac{x-1}{x+4}$ $(x\neq 3)$

$$\lim_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}=\lim_{x\to 3}\frac{x-1}{x+4}=\frac{2}{7}$$

② $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ $x=4$: $\frac{2-2}{4-4}=\frac{0}{0}$

$$\frac{\sqrt{x}-2}{x-4}=\frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}=\frac{1}{\sqrt{x}+2}\quad(x\neq 4)$$

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

③ $$\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\sec x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin x/\cos x}{1/\cos x}=\lim_{x\to\frac{\pi}{2}}\sin x=1$$

④ $\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1} \quad (x \neq 1)$

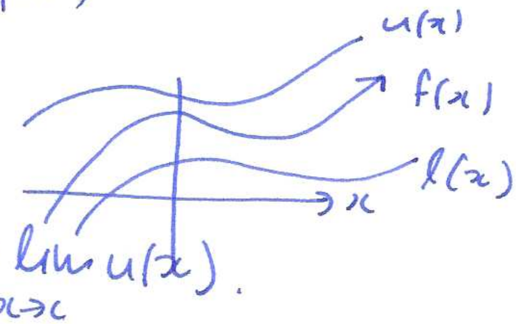
$= \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$

§2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin x}{x}$? $x=0$: get $\frac{0}{0}$ indeterminate form

A: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (try $x = 0.1, 0.01, 0.001, \ldots$)

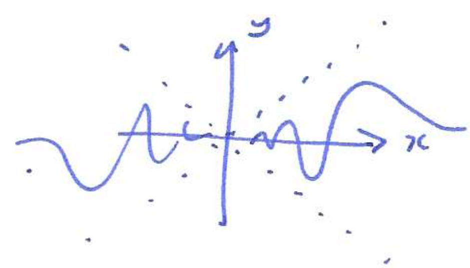
squeeze thm suppose $l(x) \le f(x) \le u(x)$



Thm suppose $l(x) \le f(x) \le u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$.

then $\lim_{x\to c} f(x) = L$

Example $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$



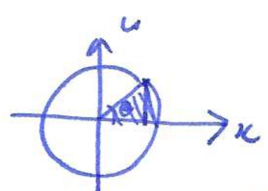
$-1 \le \sin(x) \le 1$

$-|x| \le x\sin(x) \le |x|$

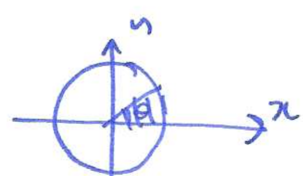
$\lim_{x\to 0} |x| = 0 \quad \lim_{x\to 0} -|x| = 0 \quad \Rightarrow \quad \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Thm $\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$

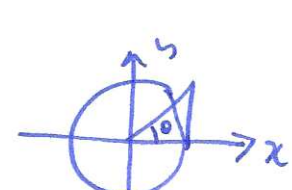
Proof (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$, consider the following three areas:



$\le$



$\le$



area of triangle $\frac{1}{2}bh$
$\frac{1}{2} \cdot 1 \cdot \sin\theta$

area of sector
$\pi r^2 \cdot \frac{\theta}{2\pi}$

area of triangle $\frac{1}{2}bh$
$\frac{1}{2} \cdot \frac{\sin\theta}{\cos\theta}$.