the special logarithm with base $b=e$ is called the natural logarithm $\ln(x)$

- inverse function properties $f^{-1}(f(x)) = x = f(f^{-1}(x))$

$$b^{\log_b x} = x = \log_b(b^x)$$

- logarithm rules: $\log_b(1) = 0$ $\quad \log_b(b) = 1$

$\log_b(st) = \log_b(s) + \log_b(t)$ $\qquad \log_b\left(\frac{1}{t}\right) = -\log_b(t)$

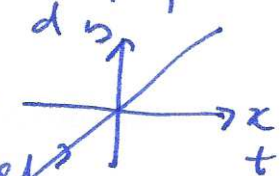
$\log_b\left(\frac{s}{t}\right) = \log_b(s) - \log_b(t)$ $\qquad \log_b(s^t) = t\log_b(s)$

base conversion: $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ for any a, so $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$

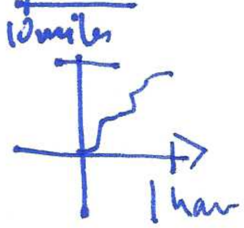
§2.1 Limits, rates of change, tangent lines

motivation: velocity $= \dfrac{\text{distance}}{\text{time}}$ $\qquad$ example: driving at constant speed

velocity = slope of line

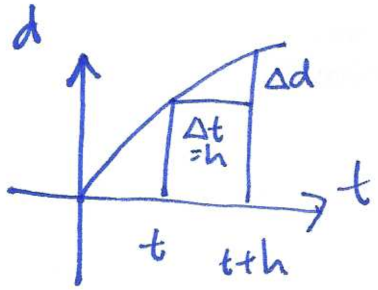


problem: what happens if you don't drive at constant speed?



average speed $= \dfrac{\text{distance travelled}}{\text{time}}$

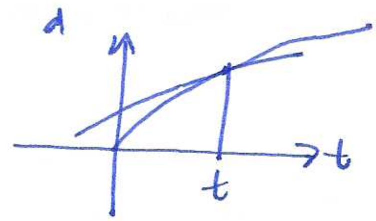
- we can look at average speed over any time interval, including very short ones.



average speed on an interval $[t, t+h]$ is

$$\frac{\Delta d}{\Delta t} = \frac{f(t+h) - f(t)}{t+h-t} = \frac{f(t+h) - f(t)}{h}$$

Q: what is the speed at time t?
(sometimes called the instantaneous speed/rate of change)



A: speed (velocity) is the slope of the tangent line at t

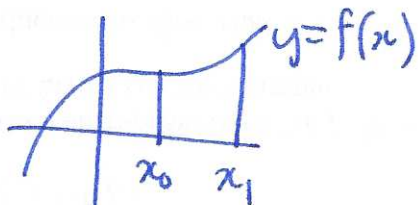
idea/hope: as the length of the interval $[t, t+h]$ gets small, the average speed gets closer to the slope of the tangent line.

this works for "nice" functions.

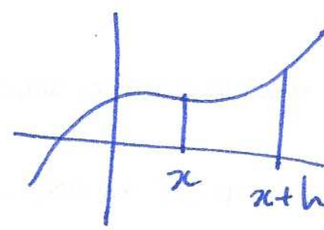
observation: this works for any function $y=f(x)$, not just distance.

summary: average rate of change over an interval $[x_0, x_1]$ is

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}$$



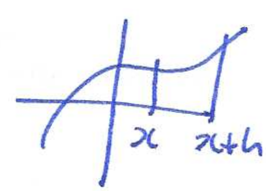
often:



§2.2 Limits

aim: want to find slopes of tangent lines. $\frac{f(x+h)-f(x)}{h}$.

know: average rate of change $\frac{f(x+h)-f(x)}{h}$



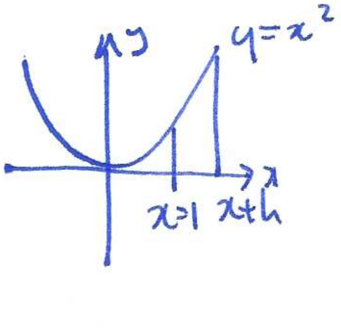
Q: why not just set $h=0$?

A: doesn't work, get $\frac{f(x)-f(x)}{0} = \frac{0}{0}$ undefined. ☹

Observations

① if we draw careful pictures, the average slope gets close to the slope of the tangent line, as the length of the interval gets small.

② seems to work for sample calculations:



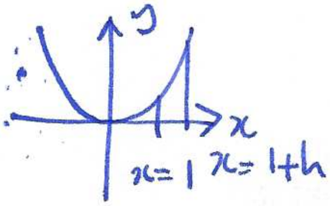
$h=1$: $\frac{f(2)-f(1)}{2-1} = \frac{4-1}{1} = 3$

$h=0.5$: $\frac{f(1.5)-f(1)}{1.5-1} = \frac{\frac{9}{4}-1}{1/2} = \frac{5}{2} = 2.5$

$h=0.1$: $\frac{f(1.1)-f(1)}{0.1} = \frac{1.21-1}{0.1} = 2.1$

$h = 0.01$ $\qquad \dfrac{1.0201 - 1}{0.01} = 2.01$

3 seems to work algebraically

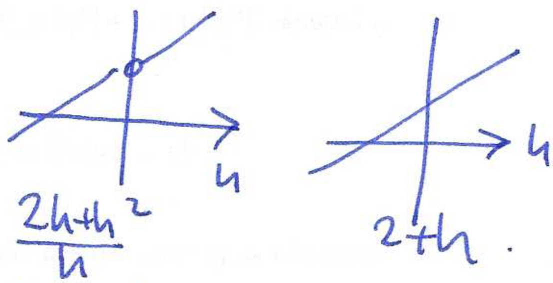


average rate of change from $x=1$ to $x+h = 1+h$

$$= \frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2 - 1^2}{h} = \frac{1+2h+h^2-1}{h}$$

$$= \frac{2h+h^2}{h} = 2+h$$

$(h \neq 0!)$

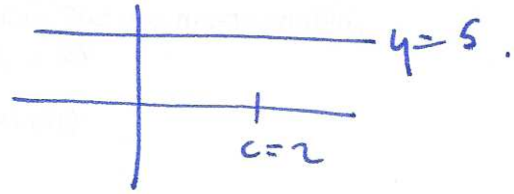


Defn Let f be a function defined on an interval containing c, but not necessarily at c. We say "the limit of $f(x)$ as x approaches c is equal to L" if $|f(x) - L|$ becomes arbitrarily small as x gets close to c

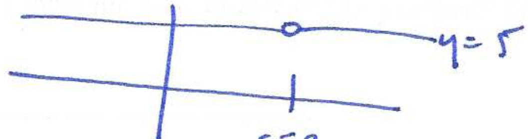
notation: $\lim_{x \to c} f(x) = L$ or $f(x) \to L$ as $x \to c$

we also say: "$f(x)$ converges to L as x tends to c"

Examples a) $f(x) = 5$, $c = 2$



b) $f(x) = \dfrac{5(x-2)}{(x-2)}$, $c = 2$



want to show: $|f(x) - 5|$ close to 0 if x close to 2

$|f(x) - 5| = |5-5| = 0$ for all $x \neq 2$, so this is true.

c) $\lim_{x \to 2} 2x+1 = 5$