$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \longleftrightarrow \quad \tan^2 x + 1 = \sec^2 x$$

$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \longleftrightarrow \quad 1 + \cot^2 x = \operatorname{cosec}^2 x$$

<u>Double angle</u> $\quad \sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

<u>Addition</u> $\quad \sin(A+B) = \sin A\cos B + \cos A \sin B$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

<u>special case</u> (shift) $\quad \sin\left(x + \frac{\pi}{4}\right) = \cos(x)$.

<u>Example</u> suppose $\sin\theta = \frac{2}{3}$, find $\cos\theta$, $\tan\theta$, $\sin 2\theta$

$\cos\theta = \frac{\sqrt{5}}{3} \quad \tan\theta = \frac{2}{\sqrt{5}} \quad \sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$.

§1.5 Inverse functions

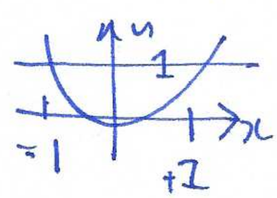
recall: $f : \mathbb{R} \to \mathbb{R}$ (domain ↗, ↖ range)
$x \mapsto f(x)$

want: the inverse function should be the reverse of this

$$\mathbb{R} \xleftarrow{f^{-1}} \mathbb{R}$$
$$f^{-1}(x) \leftarrow x$$

<u>problem</u>: the inverse is often not a function.

<u>example</u> $\quad f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$



$f(+1) = 1$
$f(-1) = 1$

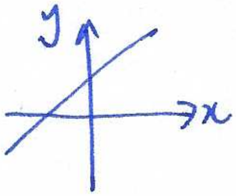
Q: what is $f^{-1}(1)$?

Q: when does a function have an inverse?

A: when it passes the <u>horizontal line test</u> (one-to-one / injective)

$\Leftrightarrow$ for each element $c \in$ range (image?), there is at most one (unique) x s.t. $f(x) = c$.

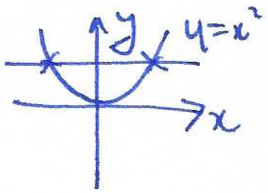
Example ~~f(x)=2x~~ $y = x+1$



Q: how do we find a formula for the inverse?

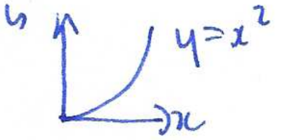
A: 1. write down $y = f(x)$
2. solve for x in terms of y, ie. $x = g(y)$
3. $f^{-1}(x) = g(x)$
4. check!

Example $f(x) = x^2$



problem: no inverse! (doesn't pass horizontal line test)

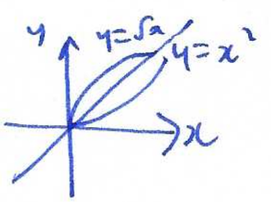
fix: restrict domain consider $f: [0,\infty) \to [0,\infty)$
$x \longmapsto x^2$



does pass the horizontal line test so has an inverse we call $\sqrt{x}$.

$f^{-1}: [0,\infty) \to [0,\infty)$
$x \longmapsto \sqrt{x}$.

How to draw the graph of the inverse



· reflect in $y=x$.

reason: graph of f is pairs $(x, f(x))$
input output

graph of f^{-1} is pairs $(f(x), x)$.

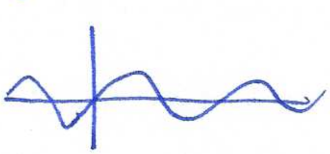
$y = f(x) \leftrightarrow f^{-1}(y) = x$

$(x, f(x)) \leftrightarrow (f^{-1}(y), y)$

swap and relabel.

Inverse trig functions

$y = \sin(x)$

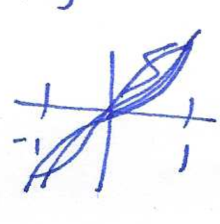


problem: not one-to-one

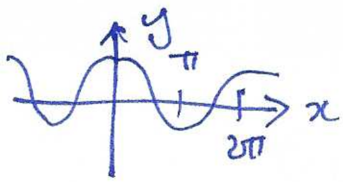
fix: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(x): [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$\sin^{-1}(x): [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$



similarly $y = \cos(x)$



restrict domain to $[0, \pi]$

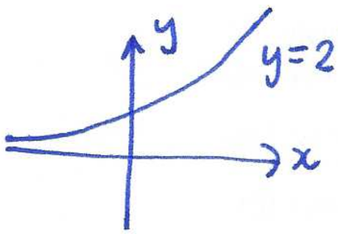
$\cos(x) : [0,\pi] \to [-1,1]$

$\cos^{-1}(x) : [-1,1] \to [0,\pi]$

§1.6 Exponential and logarithm functions

Example $x \mapsto 2^x$

x	-2	-1	0	1	2	3	4
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



can use any positive number instead of 2, $f(x) = b^x$, $b > 0$

useful properties:
- positive $(b>0)$
- b^x increasing if $b > 1$, decreasing if $b < 1$
- b^x grows faster than any polynomial

exponent rules $b^0 = 1$ $\quad b^x b^y = b^{x+y}$ $\quad b^{-x} = \frac{1}{b^x}$ $\quad \frac{b^x}{b^y} = b^{x-y}$

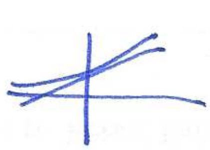
$(b^x)^y = b^{xy}$ $\quad b^{1/n} = \sqrt[n]{b}$

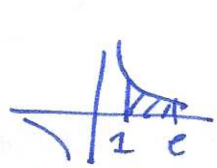
- there is a special exponential function e^x, $e = 2.71828\ldots$

Key properties of e^x

① e is the unique number s.t. e^x has slope 1 at $x = 0$

② e is the unique number s.t. the area under the curve $y = \frac{1}{x}$ between 1 and e has area 1





Logarithms the logarithm is the inverse function of the exponential function