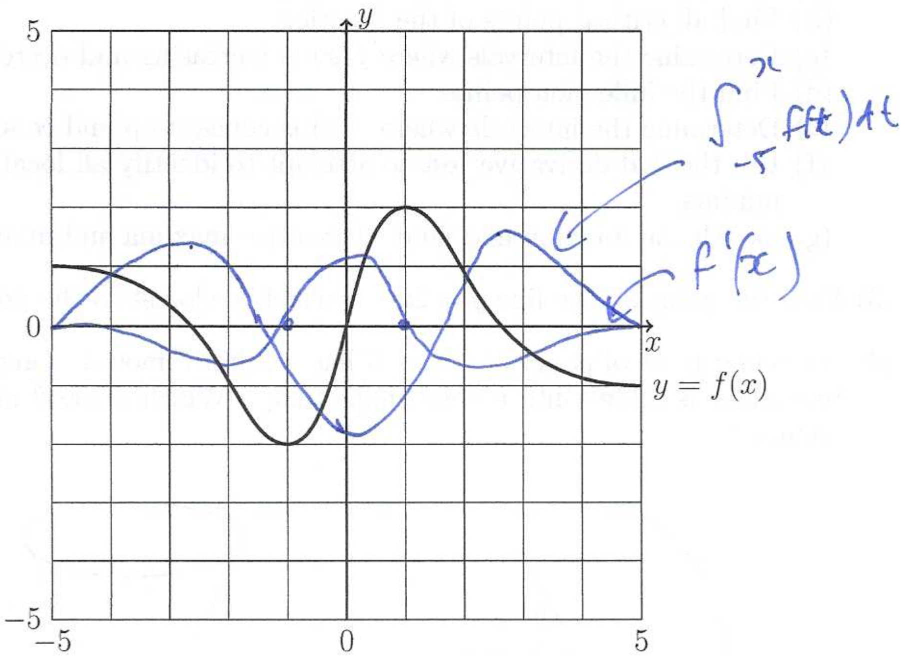
Math 231 Calculus 1 Fall 24 Sample Midterm 3

(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $[-5,-1)\cup(1,5]$

(b) Label all regions where $f'(x) > 0$. $(-1,1)$

(c) What is $\lim_{x\to\infty} f(x)$? -1

(d) What is $\lim_{x\to-\infty} f'(x)$? 0

(e) What is $\lim_{x\to\infty} f''(x)$? 0

(f) Sketch a graph of $f'(x)$ on the figure.

(g) Sketch a graph of $\int_{-5}^{x} f(t)dt$ on the figure.

(h) Label the approximate locations of all points of inflection. $-2, 0, 2$

SMT3 Solutions

Q2 $f(x) = e^{4-x^2}$

a) no vertical asymptotes. $\lim_{x\to\pm\infty} e^{4-x^2} = 0$ so left/right horizontal asymptote $y=0$

b) $f'(x) = e^{4-x^2}\cdot(-2x)$ $\quad f'(x) = 0 \Rightarrow x = 0$.

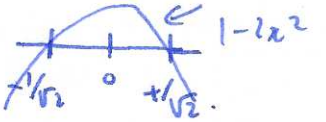
c) $f'(x) > 0$ on $(-\infty, 0)$ so f increasing on $(-\infty, 0)$

$f'(x) < 0$ on $(0, \infty)$ so f decreasing on $(0, \infty)$.

d) $f''(x) = e^{4-x^2}\cdot(-2x)^2 + e^{4-x^2}\cdot(-2) = -2e^{4-x^2}(-2x^2+1)$

$f''(x) = 0 \quad x^2 = 1/2 \quad x = \pm 1/\sqrt{2}$.

e)

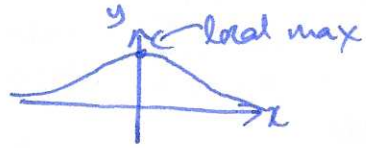


$f''(x) < 0$ on $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$ so f concave down there.

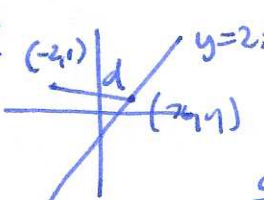
$f''(x) > 0$ on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ so f concave up there.

f) $f''(0) = -2e^4 < 0 \Rightarrow$ local max.

g)



Q3



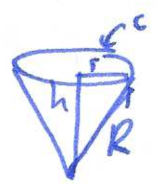
$d^2 = (x+2)^2 + (y-1)^2$ use $y = 2x-1$ $\quad d^2 = (x+2)^2 + (2x-2)^2$

$= x^2+4x+4$
$+ 4x^2 - 8x + 4$

$d^2 = 5x^2 - 4x + 8$

$\frac{d}{dx}(d^2) = 10x - 4 \qquad x = \frac{2}{5},\ y = -\frac{1}{5}$.

Q4



$A = (\pi - \frac{\theta}{2})R^2$

$c = (2\pi - \theta)R$

$c = 2\pi r \Rightarrow r = \frac{1}{2\pi}(2\pi - \theta)R = (1 - \frac{\theta}{2\pi})R$.

$r^2 + h^2 = R^2 \qquad h^2 = R^2 - r^2 = R^2 - (1-\frac{\theta}{2\pi})^2 R^2 = R^2\frac{\theta^2}{4\pi^2}$

$V = \frac{1}{3}\pi r^2 h \qquad h = \frac{R\theta}{2\pi}$

$V = \frac{1}{3}\pi\cdot(1-\frac{\theta}{2\pi})R^2\cdot\frac{R\theta}{2\pi} = \frac{R^3}{6}\theta(1-\frac{\theta}{2\pi}) = \frac{R^3}{6}(\theta - \frac{\theta^2}{2\pi})$.

$\frac{dV}{d\theta} = \frac{R^3}{6}(1 - \frac{\theta}{\pi})$ $\quad$ critical point: $\frac{dV}{d\theta} = 0$, $\theta = \pi$

Q5 a) $\lim_{x\to2}\frac{4x^2-11x+6}{3x^2-10x+8} \overset{l'H}{=} \lim_{x\to2}\frac{8x-11}{6x-10} = \frac{5}{12}$

b) $\lim_{x\to0}\frac{2x}{\tan(3x)} \overset{l'H}{=} \lim_{x\to0}\frac{2}{\sec^2(3x)\cdot3} = \frac{2}{3}$

c) $\lim_{x\to9}\frac{9-x}{\sqrt{x}-3} = \lim_{x\to9}\frac{-1}{\frac12 x^{-1/2}} = -6$

d) $\lim_{x\to2}\frac{12-3x^2}{1/2-1/x} = \lim_{x\to2}\frac{12x-3x^3}{x/2-1} \overset{l'H}{=} \lim_{x\to2}\frac{12-9x^2}{\frac12} = 6$

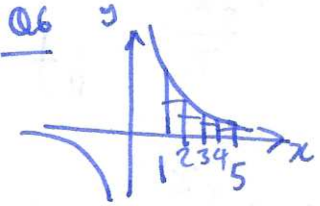
e) $\lim_{x\to0}\frac{\sin^{-1}(2x)}{\tan^{-1}(3x)} = \lim_{x\to0}\frac{\left(1/\sqrt{1-(2x)^2}\right)\cdot2}{1/(1+(3x)^2)\cdot3} = \frac{2}{3}$

f) $\lim_{x\to\infty}\frac{4-2/x+3/x^2-2/x^3}{3-2/x+11/x^3} = \frac{4}{3}$.

g) $\lim_{x\to\infty}\frac{e^{x^2}}{x^2+1} \overset{l'H}{=} \lim_{x\to\infty}\frac{e^{x^2}\cdot2x}{2x} = \lim_{x\to\infty}e^{x^2} = +\infty$.

h) $\lim_{x\to\frac12^-}\frac{\tan(\pi x)}{\ln(1-2x)} \overset{l'H}{=} \lim_{x\to\frac12^-}\frac{\sec^2(\pi x)\cdot\pi}{\frac{1}{1-2x}\cdot(-2)} = \lim_{x\to\frac12^-}\frac{\pi(1-2x)}{-2\cos^2(\pi x)} \overset{l'H}{=} \lim_{x\to\frac12^-}\frac{-2\pi}{-2\cdot2\cos(\pi x)\cdot\sin(\pi x)} = +\infty$.

Q6



$R_5 = 1\cdot\frac12+1\cdot\frac13+1\cdot\frac14+1\cdot\frac15 = \frac{77}{60}$ under estimate.

Q7 a) $\int 3x^{-1/4}+2x^{3/4}-x^{7/4}\,dx = 3x^{-5/4}\cdot\frac{4}{5} + 2x^{7/4}\cdot\frac{4}{7} - x^{11/4}\cdot\frac{4}{11}+c$

b) $\int_{-1}^{0} -x\,dx + \int_{0}^{2} x\,dx = \left[-\frac12x^2\right]_{-1}^{0} + \left[\frac12x^2\right]_{0}^{2} = \frac12+2 = \frac52$

c) $\int_1^4 2x^{-1/2}\,dx = \left[4x^{1/2}\right]_1^4 = 8-4 = 4$

d) $\int_0^3 e^{-2x}\,dx = \left[-\frac12 e^{-2x}\right]_0^3 = -\frac12 e^{-6}+\frac12$

e) $\int_0^x \frac{1}{t+2}\,dt = \left[\ln|t+2|\right]_0^x = \ln(t+2)-\ln(2)$

f) $\int\frac{1}{1+4x^2}\,dx$ $\quad u=2x$, $\frac{du}{dx}=2$ $\quad \int \frac{1}{1+u^2}\frac{dx}{du}\,du = \int\frac{1}{1+u^2}\frac12\,du = \frac12\tan^{-1}(u)+c$

$= \frac12\tan^{-1}(2x)+c$

g) $\int \frac{x}{1+3x^2}\,dx \quad u = 1+3x^2 \quad \frac{du}{dx} = 6x \quad \int \frac{x}{u}\cdot\frac{dx}{du}\,du = \int \frac{x}{u}\cdot\frac{1}{6x}\,du = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln(u)+c$

$= \frac{1}{6}\ln(1+3x^2)+C$

h) $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x)+c$

i) $\int x\cos(1+x^2)\,dx \quad u = 1+x^2 \quad \frac{du}{dx} = 2x \quad \int x\cos(u)\frac{dx}{du}\,du = \int x\cos(u)\frac{1}{2x}\,du = \int \frac{1}{2}\cos(u)\,du$

$= \frac{1}{2}\sin(u)+c = \frac{1}{2}\sin(1+x^2)+c$

j) $\int \frac{\sin(x)}{e^{\cos(x)}}\,dx \quad u = \cos x \quad \frac{du}{dx} = -\sin x \quad \int \frac{\sin x}{e^u}\frac{dx}{du}\,du = \int \frac{\sin x}{e^u}\cdot\frac{1}{-\sin x}\,du = \int -e^{-u}\,du$

$= e^{-u}+c = e^{-\cos x}+c$

k) $\int \frac{\cos x}{\sin^2 x}\,dx \quad u = \sin x \quad \frac{du}{dx} = \cos x \quad \int \frac{\cos x}{u^2}\cdot\frac{dx}{du}\,du = \int \frac{\cos x}{u^2}\cdot\frac{1}{\cos x}\,du = \int u^{-2}\,du = -u^{-1}+c$

$= \frac{-1}{\sin x}+c = -\csc x+c$

Q8 $F(x) = \int_0^x e^{-t}\sin(t)\,dt \Rightarrow F'(x) = e^{-x}\sin(x)$

$\frac{d}{dx}F(x^2) = F'(x^2)\cdot 2x = e^{-x^2}\sin(x^2)\cdot 2x$

Q9 $v(t) = (t+1)^{-3}$

$x(t) = -\frac{1}{2}(t+1)^{-2}+c \qquad x(0)=0: \quad -\frac{1}{2}+c=0 \quad c=\frac{1}{2}$

$x(t) = \frac{1}{2}-\frac{1}{2}\frac{1}{(t+1)^2} \qquad \lim_{t\to\infty} x(t) = \frac{1}{2}$, does not reach $x=10$.