Math 231 Calculus 1 Fall 24 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $f(x) = x^2 \ln(x)$.

$$2x \ln(x) + x^2 \frac{1}{x} = 2x \ln(x) + x$$

(b) $f(x) = \dfrac{\cos(x)}{e^x}$.

$$\frac{-e^x \sin x - e^x \cos x}{e^{2x}} = \frac{-\sin x - \cos x}{e^x}$$

(2) (10 points) Find the derivative of the function $f(x) = \sin^{-1}(3 - x^2)$.

$$\frac{1}{\sqrt{1+(3-x^2)^2}} \cdot (-2x)$$

(3) (10 points) Find the second derivative of the function $f(x) = \sqrt{3 - 2x}$.

$$f'(x) = \frac{1}{2}(3-2x)^{-1/2} \cdot (-2) = -(3-2x)^{-1/2}.$$

$$f''(x) = \frac{1}{2}(3-2x)^{-3/2}(-2)$$

(4) (10 points) Use implicit differentiation to find the tangent line to the curve given by the equation $3x^2y^2 + y^3 = 13$ at the point $(2, -1)$.

$$6xy^2 + 3x^2 \cdot 2y\,y' + 3y^2y' = 0$$

$$y'(6x^2y + 3y^2) = -6xy^2$$

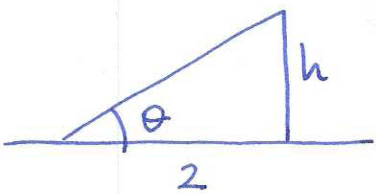
$$y' = \frac{-6xy^2}{6x^2y + 3y^2} \quad \text{at } (2,-1) \quad y' = \frac{-12}{-24+3} = \frac{12}{21} = \frac{4}{7}$$

$$y + 1 = \frac{12}{21}(x-2) = \frac{4}{7}(x-2)$$

(5) Find the following limit: $\lim_{x \to 0} \dfrac{1 + 2\sin(x) - e^{2x}}{1 - \cos(3x)}$

$$\overset{l'H}{=} \lim_{x \to 0} \frac{2\cos x - 2e^{2x}}{3\sin(3x)} \overset{l'H}{=} \lim_{x \to 0} \frac{-2\sin x - 4e^{2x}}{9\cos(3x)} = \frac{-4}{9}$$

(6) (10 points) A rocket is launched from ground level vertically upwards from a distance of 2km away. When you see it an angle of $\pi/4$ radians, the angle is increasing at a rate of 0.1 radians/sec. How fast is the rocket going up?



$$\frac{h}{2} = \tan\theta$$

$$\frac{1}{2}\frac{dh}{dt} = \sec^2\theta \, \frac{d\theta}{dt}$$

$$\frac{dh}{dt} = 2\underbrace{\sec\left(\frac{\pi}{4}\right)^2}_{2}\underset{0.1}{\cancel{\frac{\pi}{4}}} = \underset{0.4}{}\ \text{km/sec}$$

(7) (10 points) Use linear approximation to estimate $\sqrt{62}$. What is the percentage error in your approximation?

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$

$f(64-2) \approx f(64) + f'(64)(-2)$

$= 8 + \frac{1}{2}\frac{1}{8}(-2)$

$= 7\frac{7}{8}$

percentage error $= \dfrac{\left|\sqrt{62} - 7\frac{7}{8}\right|}{\sqrt{62}} \times 100 \approx 0.01260\ \%$

(8) Find the critical points for the function $f(x) = x^3 - 48x$ and use the second derivative test to classify them.

$f'(x) = 3x^2 - 48 = 3(x^2 - 16)$

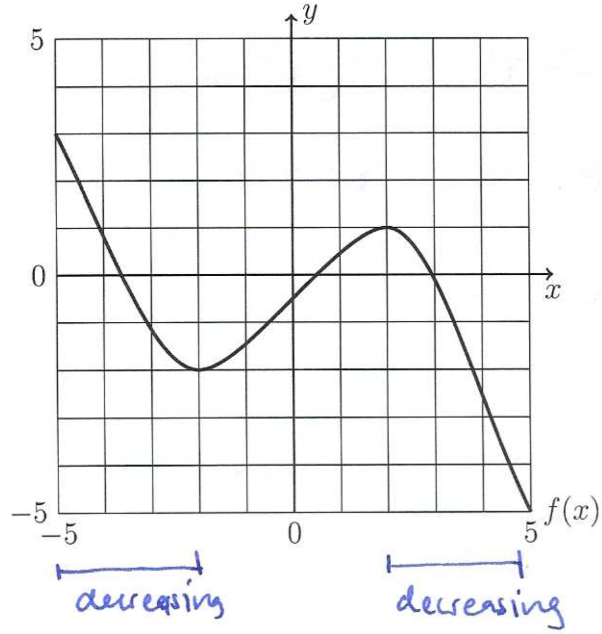
critical points : $f'(x) = 0$ $\quad x = \pm 4$.

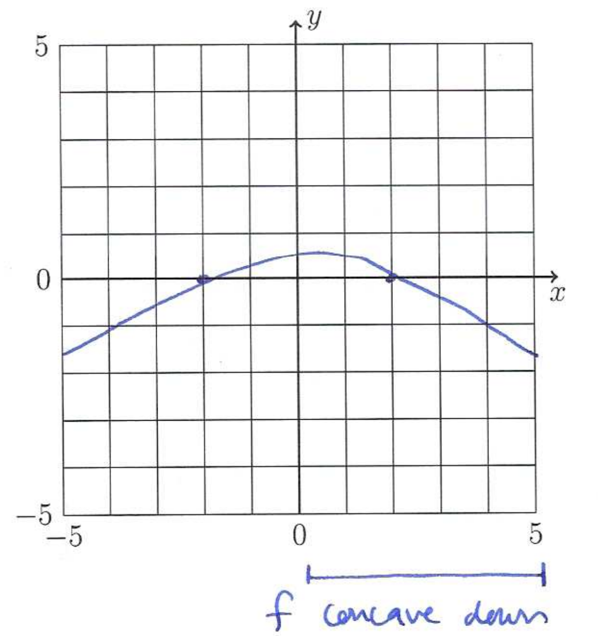
$f''(x) = 6x$

$f''(-4) = -24 < 0$ local max

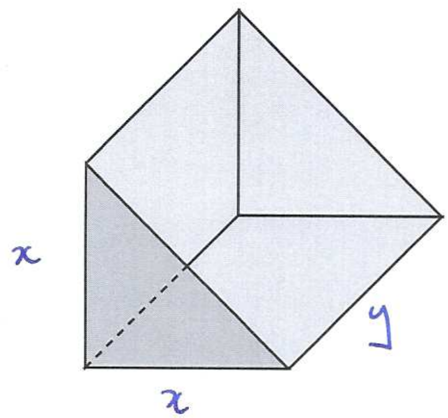
$f''(4) = 24 > 0$ local min

(9) (10 points) The graph of the function $f(x)$ is shown below. On the top set of axes mark where $f(x)$ is decreasing. On the lower set of axes sketch $f'(x)$, and then use this to find where $f(x)$ is concave down.





(10) The shovel part of a mechanical digger is to be made of metal in the shape of two rectangles welded together at right angles, supported at each end by equilateral right angled triangles, as shown below. If the total volume of the shape is 2m^3, find the minimum area of metal required.



$$V = \frac{1}{2}x^2 y = 2 \qquad y = \frac{4}{x^2}.$$

$$A = 2xy + x^2$$

$$A = 2x \cdot \frac{4}{x^2} + x^2 = \frac{8}{x} + x^2$$

$$\frac{dA}{dx} = -\frac{8}{x^2} + 2x$$

critical point $\frac{dA}{dx} = 0$

$$2x = +\frac{8}{x^2}$$

$$x^3 = 4$$

$$x = \sqrt[3]{2} = 2^{1/3}.$$

$$y = \frac{4}{\sqrt[3]{4}} = 2^2 . 2^{-2/3} = 2^{4/3}.$$

$$A = \cancel{2}\,\frac{8}{\sqrt[3]{2}} + \left(\sqrt[3]{2}\right)^2.$$