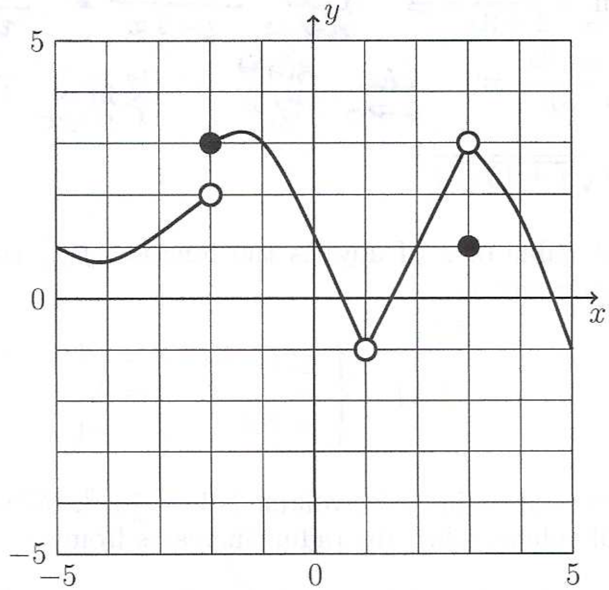
Math 231 Calculus 1 Fall 24 Sample Midterm 1

Solutions

(1) The graph of $y = f(x)$ is shown below. Evaluate each limit, or write DNE if the limit does not exist. No justifications are necessary.



(a) $\lim_{x \to -2-} f(x)$ 2

(b) $\lim_{x \to -2+} f(x)$ 3

(c) $\lim_{x \to -2} f(x)$ DNE

(d) $\lim_{x \to 1-} f(x)$ −1

(e) $\lim_{x \to 1+} f(x)$ −1

(f) $\lim_{x \to 3} f(x)$ 3

(2) Evaluate these limits. For an infinite limit, write $+\infty$ or $-\infty$. If a limit does not exist (DNE), you must justify why this is the case.

(a) $\lim_{x\to -2} \dfrac{x+2}{x^2-x-6}$ $= \lim_{x\to -2} \dfrac{(x+2)}{(x+2)(x-3)} = \lim_{x\to -2} \dfrac{1}{x-3} = -\dfrac{1}{5}$

(b) $\lim_{x\to -\infty} \dfrac{\sqrt{3+2x^2}}{2+3x}$ $= \lim_{x\to \infty} \dfrac{\sqrt{3+2x^2}}{2-3x} = \dfrac{\sqrt{3/x^2+2}}{2/x-3} = \dfrac{\sqrt{2}}{-3}$.

$\theta = 4x$ (c) $\lim_{x\to 0} \dfrac{\sin 4x}{5x}$ $= \lim_{\theta\to 0} \dfrac{\sin\theta}{\frac{5}{4}\theta} = \dfrac{4}{5}\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = \dfrac{4}{5}$

(d) $\lim_{x\to 0} \dfrac{x}{\sqrt{x+16}-4}$

(3) For what value of c (if any) is the function $f(x)$ continuous at $x=-1$? Justify your answer.

$$f(x) = \begin{cases} \frac{1}{x-3} - 2x & x < -1 \\ c & x = 2 \\ \frac{\cos(\pi x)}{x} & x > -1 \end{cases}$$

(4) For a sphere of radius r, its volume is $V = \frac{4}{3}\pi r^3$. What is the average rate of change of volume when the radius increases from $r=2$ to $r=3$?

(5) Show that $e^x = 4 - x^2$ has a solution for some $x > 0$. You do not need to find this solution.

Q2 d) $= \lim_{x\to 0} \dfrac{x}{\sqrt{x+16}-4} \cdot \dfrac{\sqrt{x+16}+4}{\sqrt{x+16}+4} = \lim_{x\to 0} \dfrac{x(\sqrt{x+16}+4)}{x+16-16} = \lim_{x\to 0} \sqrt{x+16}+4 = 8$

Q3

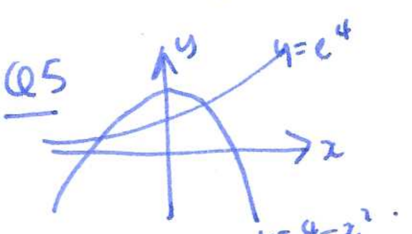
$\lim_{x\to -1^-} \dfrac{1}{x-3} - 2x = \dfrac{1}{-4} + 2 = \dfrac{7}{8}$

$\lim_{x\to -1^+} \dfrac{\cos(\pi x)}{x} = \dfrac{\cos(-\pi)}{-1} = 1$

$\frac{7}{8} \neq 1$ so no value of c makes $f(x)$ cts.

Q4 $\dfrac{V(3)-V(2)}{3-2} = \dfrac{\frac{4}{3}\pi 27 - \frac{4}{3}\pi \cdot 8}{1} = \frac{4}{3}\pi \cdot 19$.

Q5

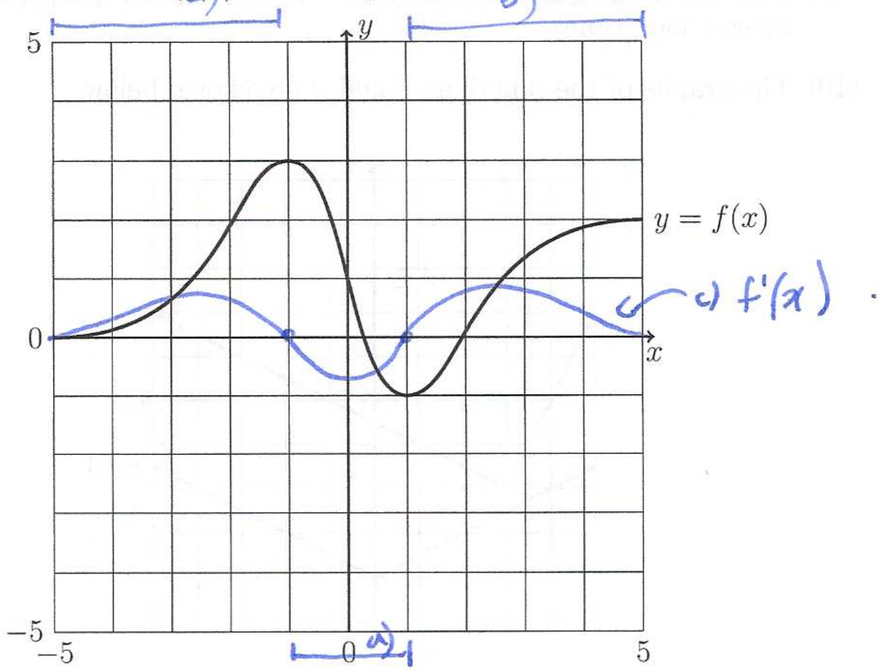


consider $f(x) = 4 - x^2 - e^x$ $f(0) = 3 > 0$

$f(2) = -e^2 < 0$

so function is cts goes from +ve at 0 to −ve at 2 so has solution in $(0,2)$

(6) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $(-1,1)$

(b) Label all regions where $f'(x) > 0$. $[-5,-1) \cup (1,5]$.

(c) Sketch a graph of $f'(x)$ on the figure.

(d) What is $\lim_{x\to\infty} f(x)$? 0

(e) What is $\lim_{x\to-\infty} f'(x)$? 0

(7) Use the limit definition of the derivative to evaluate $f'(2)$, where

(a) $f(x) = \dfrac{1}{x-3}$

(b) $f(x) = \dfrac{1}{\sqrt{1+x}}$

(8) Find the derivatives of the following functions

(a) $2x^4 - 3x^2 + \sqrt{x^5} - 3\sqrt[5]{1/x^2}$

(b) $4x^3e^x$

(c) $\dfrac{5x+2}{4-3x}$

(d) $\dfrac{5-\sqrt{2x}}{2-\cos(x)}$

(e) $\tan(x)$

(f) $\sin^2(x)$

(g) $x^4e^x\cos(x)$

(h) $\dfrac{\sqrt{x+e^x}}{1+\cos^{-1}(3x)}$

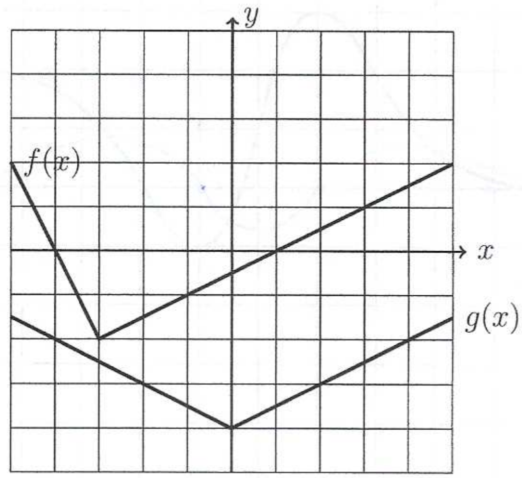
(i) x^{x^2}

(j) $\sqrt{\ln(\sec(x))}$

(k) $\cos^{-1}(3x-2)$

(9) Find the second derivatives of the functions above (skip (d, j, h, k) as they takes a long time).

(10) The graphs of the functions f and g are shown below.



(a) Let $h(x) = f(x)g(x)$ Find $h'(3)$.
(b) Let $h(x) = f(x)/g(x)$. Find $h'(-1)$.
(c) Let $h(x) = f(g(x))$. Find $h'(2)$.

a) $h'(3) = f'(3)g(3) + f(3)g'(3) = \frac{1}{2}\left(-2\frac{1}{2}\right) + 1\frac{1}{2} = \frac{-5}{4} + \frac{1}{2} = \frac{-3}{4}$.

b) $h'(-1) = \frac{g(-1)f'(-1) - f(-1)g'(-1)}{g(-1)^2} = \frac{(-7/2)\frac{1}{2} - (-1)(-\frac{1}{2})}{(-7/2)^2} = \frac{-7/4 - \frac{1}{2}}{49/4} = \frac{-9/4}{49/4}$

$= -9/49$.

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Q7 a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h-3} - \frac{1}{x-3}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{(x-3)-(x+h-3)}{(x+h-3)(x-3)}$

$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x+h-3)(x-3)} = \lim_{h\to 0} \frac{-1}{(x+h-3)(x-3)} = \frac{-1}{(x-3)^2}$. at $x=2$: $\frac{-1}{(-1)^2} = -1$

b) $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{3+h}} - \frac{1}{\sqrt{3}}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{\sqrt{3}-\sqrt{3+h}}{\sqrt{3+h}\,\sqrt{3}}$

$= \lim_{h\to 0} \frac{1}{h} \frac{\sqrt{3}-\sqrt{3+h}}{\sqrt{3+h}\,\sqrt{3}} \cdot \frac{\sqrt{3}+\sqrt{3+h}}{\sqrt{3}+\sqrt{3+h}} = \lim_{h\to 0} \frac{3-(3+h)}{h\sqrt{3}\sqrt{3+h}(\sqrt{3}+\sqrt{3+h})} = \lim_{h\to 0} \frac{-h}{h\sqrt{3}\sqrt{3+h}(\sqrt{3}+\sqrt{3+h})}$

$= \lim_{h\to 0} \frac{-1}{\sqrt{3}\sqrt{3+h}(\sqrt{3}+\sqrt{3+h})} = \frac{-1}{3 \cdot 2\sqrt{3}}$.

Q8 a) $f'(x) = 8x^3 - 6x + \frac{5}{2}x^{3/2} + 3 \cdot -\frac{2}{5} x^{-7/5}$

b) $f(x) = 4x^3e^x \quad f'(x) = (4x^3)'e^x + 4x^3(e^x)' = 12x^2e^x + 4x^3e^x$

c) $\frac{(4-3x)(5x+2)' - (4-3x)'(5x+2)}{(4-3x)^2} = \frac{(4-3x) \cdot 5 + 3(5x+2)}{(4-3x)^2} = \frac{26}{(4-3x)^2} = \frac{26}{16-24x+9x^2}$

d) $\frac{(2-\cos(x))(5-\sqrt{2x})' - (2-\cos(x))'(5-\sqrt{2x})}{(2-\cos(x))^2} = \frac{(2-\cos(x))(-\sqrt{2}\cdot\frac{1}{2}x^{-1/2}) - \sin(x)(5-\sqrt{2x})}{(2-\cos(x))^2}$.

e) $\tan(x) = \frac{\sin x}{\cos x} \quad (\tan x)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$.

f) $(\sin(x).\sin(x))' = (\sin x)'\sin x + \sin x(\sin x)' = 2\sin x\cos x$.

g) $(x^4e^x\cos x)' = (x^4e^x)'\cos x + x^4e^x(\cos x)'$

$= (4x^3e^x + x^4e^x)\cos x - x^4e^x\sin x$.

Q9 a) $f''(x) = 24x^2 - 6 + \frac{15}{4}x^{1/2} + \frac{42}{25}x^{-12/5}$

b) $24xe^x + 12x^2e^x + 12x^2e^x + 4x^3e^x$

c) $\frac{(4-3x)^2 \cdot (26)' - 26(-24+18x)}{(4-3x)^4} = \frac{26(24-18x)}{(4-3x)^4}$.

e) $f'(x) = (\sec x)(\sec x)$ $\quad f''(x) = (\sec x)'\sec x + (\sec x)(\sec x)' = 2\sec x . \sec x \tan x$

$= 2\sec^2 x \tan x$.

f) $f'(x) = 2\sin x \cos x$ $\quad f''(x) = 2(\sin x)'\cos x + 2\sin x(\cos x)' = 2\cos^2 x - 2\sin^2 x$.

g) $f'(x) = 4x^3 e^x \cos x + x^4 e^x \cos x - x^4 e^x \sin x$.

$f''(x) = 12x^2 e^x \cos x + 4x^3 e^x \cos x + 4x^3 e^x (-\sin x)$

$+ 4x^3 e^x \cos x + x^4 e^x \cos x + x^4 e^x (-\sin x)$

$- 4x^3 e^x \sin x - x^4 e^x \sin x - x^4 e^x \cos x$.