Math 231 Calculus 1 Fall 24 Final b

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without CAS capabilities, and a US Letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $2\sqrt[3]{x} - \dfrac{3}{\sqrt{x}} - 4x^3$

$\frac{2}{3}x^{-2/3} + \frac{3}{2}x^{-3/2} - 12x^2$

(b) $x^2\sin(x)$

$2x\sin(x) + x^2\cos(x)$

(2) (10 points) Find the derivative of the following functions.

(a) $\dfrac{2 - \ln(x)}{e^x}$

$$\frac{e^x\left(-\frac{1}{x}\right) - (2 - \ln(x)) \cdot e^x}{(e^x)^2}$$

(b) $\sin^{-1}(3x + 1)$

$$\frac{1}{\sqrt{1-(3x+1)^2}} \cdot 3$$

(3) (10 points) Find the derivative of the following functions.

(a) $\sqrt{\tan(\ln(x))}$

$$\frac{1}{2}\left(\tan(\ln(x))\right)^{-1/2} \cdot \frac{1}{\cancel{1+(\ln(x))^2}} \cdot \frac{1}{x}$$

$$\sec^2(\ln(x))$$

(b) $xy^2 - 3y^3 = e^x$ (Use implicit differentiation to find y' implicitly.)

$$y^2 + x\,2yy' - 9y^2y' = e^x$$

$$y'(2xy - 9y) = e^x - y^2$$

$$y' = \frac{e^x - y^2}{2xy - 9y}$$

(4) (10 points)

(a) State the definition of $f'(x)$ as a limit.

(b) Use the limit definition of the derivative to find the derivative of $f(x) = \frac{1}{x}$. Do *not* use L'Hôpital's rule.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b) $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-h}{hx(x+h)}$

$= \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$

(5) (10 points) Consider $f(x) = 4 - 3x^2 - x^3$.

(a) Find the derivative for $f(x)$, and the critical points.

$x = 0, -2$

$f'(x) = -6x - 3x^2 = -3x(\cancel{x}\, 2 + x)$

(b) Find the equation of the tangent line at $x = 1$.

$f'(1) = -9$

$f(1) = 4 - 3 - 1 = 0$

$y = -9(x-1)$

(c) Find the intervals for which $f(x)$ is decreasing.

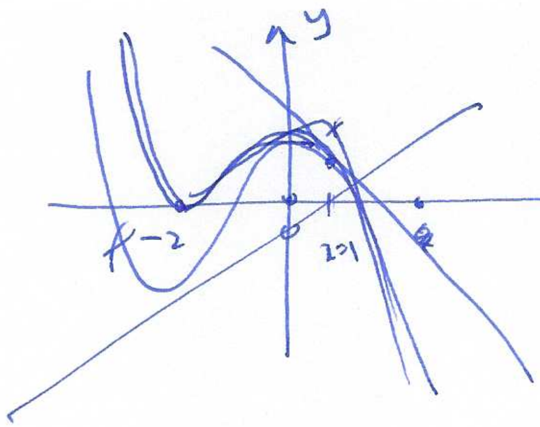
$(-\infty, -2) \cup (\cancel{4}\, 0, \infty)$

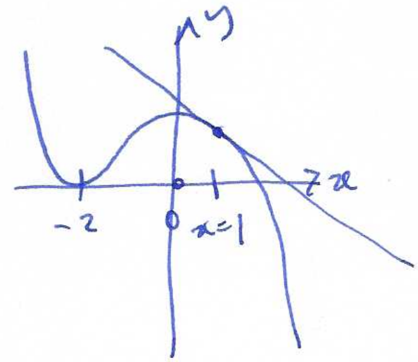
(d) Find the intervals for which $f(x)$ is concave up.

$f''(x) = -6 - 6x = -6(x+1)$

$(-\infty, -1)$

(e) Sketch the graph of $f(x)$, and the tangent line at $x = 1$.





(6) (10 points) A leak from an oil tank on a straight shore forms a semicircular oil slick. If the area of the oil slick grows at $3\text{m}^2/\text{min}$, how fast is the radius increasing when the radius is 4m?

$$A = \frac{1}{2}\pi r^2$$

$$\underbrace{\frac{dA}{dt}}_{3} = \underset{\uparrow \atop 4}{\pi r}\,\frac{dr}{dt} \qquad \frac{dr}{dt} = \frac{3}{4\pi} \text{ m/min}$$

(7) (10 points) Find the following limts. Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x\to 2} \frac{3x^2-5x-2}{x^2-5x+6}$. $\overset{l'H}{=} \lim_{x\to 2} \frac{6x-5}{2x-5} = \frac{7}{-1}$

(b) Find $\lim_{x\to 0} \frac{\sin 5x}{2x}$. $\overset{l'H}{=} \lim_{x\to 0} \frac{5\cos(5x)}{2} = \frac{5}{2}$

(c) Find $\lim_{x\to+\infty} \frac{4x-3x^2}{e^{2x}}$. $\overset{l'H}{=} \lim_{x\to\infty} \frac{4-6x}{2e^{2x}} \overset{l'H}{=} \lim_{x\to\infty} \frac{-6}{4e^{4x}} = 0$

(8) (10 points) Evaluate the following integrals.

(a) $\int \left(3x^2 - 2\sqrt[3]{x} - \frac{4}{x} - \frac{1}{\sqrt{x}}\right) dx$

$x^3 - \frac{6}{4} x^{4/3} - 4\ln|x| - 2x^{1/2} + c$

(b) $\int_0^2 e^{6x}\, dx$

$u = 6x$

$\frac{du}{dx} = 6$

$\int_0^{12} e^u \frac{dx}{du}\, du = \frac{1}{6}\int_0^{12} e^u\, du$

$\frac{1}{6}\left[e^u\right]_0^{12} = \frac{1}{6}\left(e^{12} - 1\right)$

(9) (10 points) Evaluate the following integrals.

(a) $\int x\cos(2x^2)\,dx$

$u = 2x^2$

$\frac{du}{dx} = 4x$

$\int x\cos(u)\frac{dx}{du}\,du = \int x\cos(u)\frac{1}{4x}\,du = \frac{1}{4}\sin(u) + c$

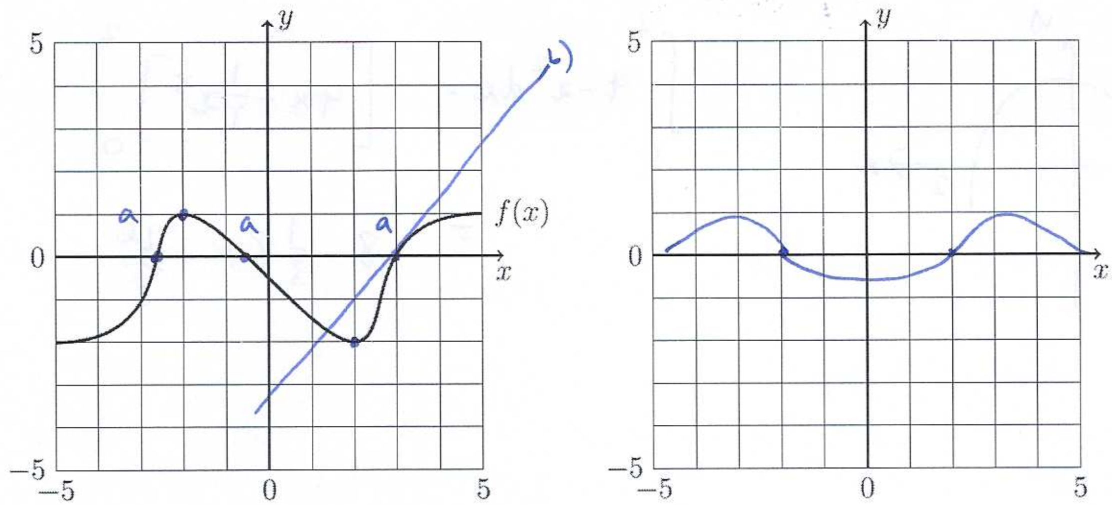
$= \frac{1}{4}\sin(2x^2) + c$

(b) If $\int_0^7 f(x)\,dx = 6$ and $\int_2^7 f(x)\,dx = -4$, find $\int_0^2 f(x)\,dx$.

$\int_0^2 f(x)\,dx + \underbrace{\int_2^7 f(x)\,dx}_{-4} = \underbrace{\int_0^7 f(x)\,dx}_{6}$

$\int_0^2 f(x)\,dx = 10$

(10) (10 points) Consider the function $f(x)$ determined by the graph below.



(a) Label the roots of $f(x)$ on the graph above. $-2\frac{1}{2}, -\frac{1}{2}, 3.$

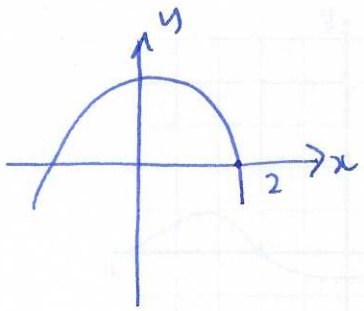
(b) On the graph above, sketch the tangent line at $x = 3$.

(c) List all the critical points of $f(x)$. $-2, 2$

(d) Sketch $y = f'(x)$ on the right hand graph.

(e) Estimate the intervals where $f(x)$ is concave down. $(-3,0) \cup (3\frac{1}{2}, 5)$

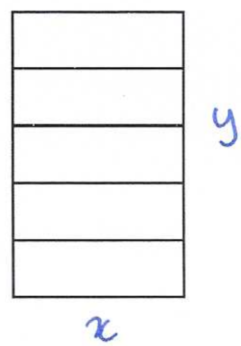
(11) (10 points) Find the area below the graph $f(x) = 4 - x^2$ which lies in the first quadrant.



$$\int_0^2 4 - x^2 \, dx = \left[4x - \frac{1}{3}x^3\right]_0^2$$

$$= 8 - \frac{1}{3} 8 = \frac{16}{3}$$

(12) (10 points) You wish to build a rectangular bookcase with five shelves, as shown below. If you have 36m of planks, what are the dimensions of the largest area bookshelf you can make?



$A = xy$

$L = 6x + 2y = 36$

$y = 18 - 3x$

$A = x(18-3x) = 18x - 3x^2$

$\frac{dA}{dx} = 18 - 6x$

$\frac{dA}{dx} = 0 \Rightarrow x = 3$

$y = 9$