Math 231 Calculus 1 Fall 24 Final a

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without CAS capabilities, and a US Letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

(1) (10 points) Find the derivative of the following functions.

(a) $2x^3 - 4\sqrt[3]{x} - \dfrac{2}{\sqrt{x}}$

$6x^2 - \frac{4}{3}x^{-2/3} + x^{-3/2}$

(b) $x^2\cos(x)$

$2x\cos(x) + x^2 \cdot -\sin(x)$

(2) (10 points) Find the derivative of the following functions.

(a) $\dfrac{e^x}{3 - \ln(x)}$

$$\frac{(3-\ln(x))e^x - \left(-\frac{1}{x}\right)e^x}{(3-\ln(x))^2}$$

(b) $\sin^{-1}(2x+1)$

$$\frac{1}{\sqrt{1-(2x+1)^2}} \cdot 2$$

(3) (10 points) Find the derivative of the following functions.

(a) $\tan\left(\sqrt{\ln(x)}\right)$

$$\sec^2\left(\sqrt{\ln(x)}\right) \cdot \cancel{\frac{1}{1+\ln(x)}} \cdot \frac{1}{2}\ln(x)^{-1/2} \cdot \frac{1}{x}$$

(b) $2y^3 - xy^2 = e^x$ (Use implicit differentiation to find y' implicitly.)

$$6y^2y' - y^2 - x \cdot 2yy' = e^x$$

$$y'(6y^2 - 2xy) = e^x + y^2$$

$$y' = \frac{e^x + y^2}{6y^2 - 2xy}$$

(4) (10 points)
(a) State the definition of $f'(x)$ as a limit.
(b) Use the limit definition of the derivative to find the derivative of $f(x) = \frac{1}{x}$. Do *not* use L'Hôpital's rule.

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b) $\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-h}{hx(x+h)}$

$= \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$

(5) (10 points) Consider $f(x) = x^3 - 3x^2 - 2$.

(a) Find the derivative for $f(x)$, and the critical points.

critical points $x = 0, 2$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

(b) Find the equation of the tangent line at $x = 1$.

$f'(1) = -\cancel{8}3$

$f(1) = 1 - 3 - 2 = -4$

$y + 4 = -3(x-1)$

(c) Find the intervals for which $f(x)$ is increasing.

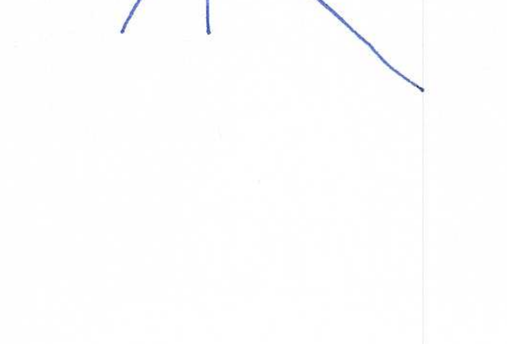
$(-\infty, 0) \cup (2, \infty)$

(d) Find the intervals for which $f(x)$ is concave down.

$f''(x) = 6x - 6$
$= 6(x-1)$

$(-\infty, 1)$

(e) Sketch the graph of $f(x)$, and the tangent line at $x = 1$.



(6) (10 points) A leak from an oil tank on the edge of a straight shore forms a semicircular oil slick. If the area of the oil slick grows at $6\text{m}^2/\text{min}$, how fast is the radius increasing when the radius is 5m?

$$A = \frac{1}{2}\pi r^2$$

$$\underbrace{\frac{dA}{dt}}_{6} = \pi \underset{\substack{\uparrow \\ 5}}{r} \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{6}{5\pi} \text{ m/min}$$

(7) (10 points) Find the following limts. Note: the possible answers for limits are a number, $+\infty$, $-\infty$ or "does not exist" (DNE). Justify your answers.

(a) Find $\lim_{x \to 2} \frac{x^2 - 5x + 6}{3x^2 - 5x - 2}$. $\overset{l'H}{=} \lim_{x \to 2} \frac{2x - 5}{6x - 5} = \frac{-1}{7}$

(b) Find $\lim_{x \to 0} \frac{3x}{\sin 4x}$. $\overset{l'H}{=} \lim_{x \to 0} \frac{3}{4\cos 4x} = \frac{3}{4}$

(c) Find $\lim_{x \to +\infty} \frac{3x - 4x^2}{e^{3x}}$. $\overset{l'H}{=} \lim_{x \to +\infty} \frac{3 - 8x}{3e^{3x}} \overset{l'H}{=} \lim_{x \to +\infty} \frac{-8}{9e^{3x}} = 0.$

(8) (10 points) Evaluate the following integrals.

(a) $\displaystyle\int \left(3x^2 - 4\sqrt[3]{x} + \frac{3}{x} - \frac{1}{\sqrt{x}}\right) dx$

$x^3 - \cancel{4}.3\, x^{4/3} + 3\ln|x| - 2x^{1/2} + c$

(b) $\displaystyle\int_0^3 e^{4x}\, dx$

$u = 4x$

$\frac{du}{dx} = 4.$

$\displaystyle\int_0^{12} e^u \frac{dx}{du} du = \frac{1}{4}\int_0^{12} e^u du = \frac{1}{4}(e^{12} - 1)$

(9) (10 points) Evaluate the following integrals.

(a) $\displaystyle\int x\sin(3x^2)\,dx$

$u = 3x^2$

$\frac{du}{dx} = 6x$

$\int x \sin(u) \frac{dx}{du}\, du = \int x\sin(u) \frac{1}{6x}\, du = \frac{1}{6}\int \sin(u)\,du$

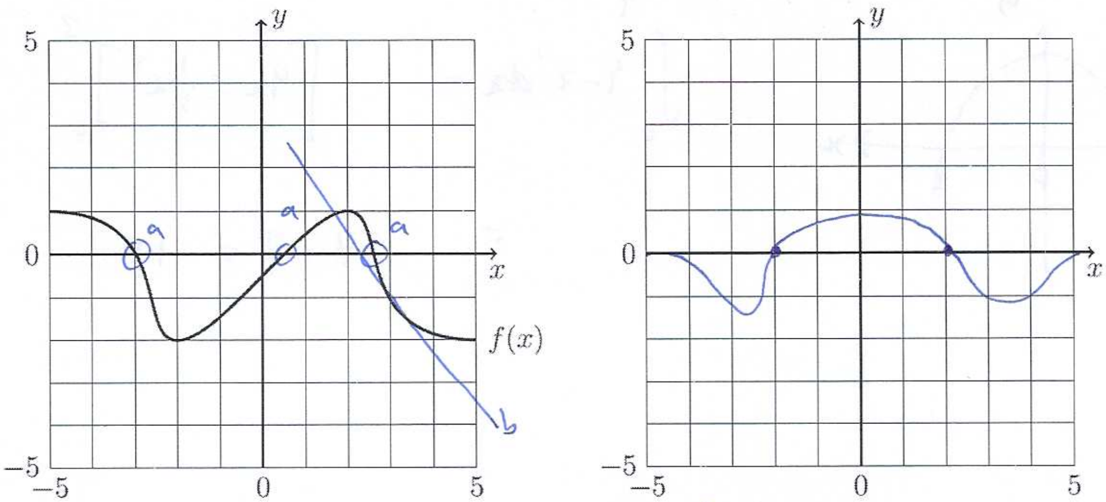
$= -\frac{1}{6}\cos(u) + c = -\frac{1}{6}\cos(3x^2) + c$

(b) If $\displaystyle\int_0^{10} f(x)\,dx = 4$ and $\displaystyle\int_3^{10} f(x)\,dx = -2$, find $\displaystyle\int_0^3 f(x)\,dx$.

$\int_0^3 f(x)\,dx + \underbrace{\int_3^{10} f(x)\,dx}_{-2} = \underbrace{\int_0^{10} f(x)\,dx}_{4}$

$\Rightarrow \int_0^3 f(x)\,dx = 6$

(10) (10 points) Consider the function $f(x)$ determined by the graph below.



(a) Label the roots of $f(x)$ on the graph above. $-3, \frac{1}{2}, 2\frac{1}{2}$

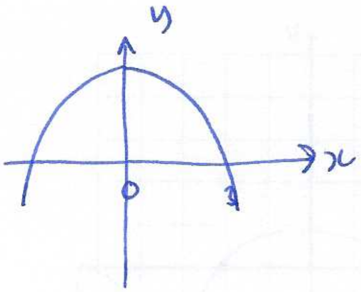
(b) On the graph above, sketch the tangent line at $x = 3$.

(c) List all the critical points of $f(x)$. $-2, 2$

(d) Sketch $y = f'(x)$ on the right hand graph.

(e) Estimate the intervals where $f(x)$ is concave up. $(-2\frac{1}{2}, 0) \cup (3\frac{1}{2}, 5)$

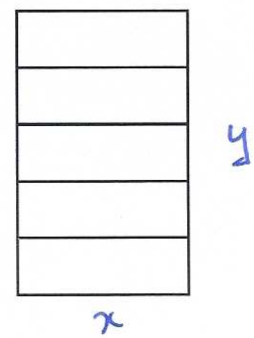
(11) (10 points) Find the area below the graph $f(x) = 9 - x^2$ which lies in the first quadrant.



$$\int_0^3 9-x^2\,dx = \left[9x - \frac{1}{3}x^3\right]_0^3$$

$$= 27 - 9 = 18$$

(12) (10 points) You wish to build a rectangular bookcase with five shelves, as shown below. If you have 24m of planks, what are the dimensions of the largest area bookshelf you can make?



$A = xy \Rightarrow$ [illegible]

$L = 6x + 2y = 24$

$\Rightarrow \quad y = 12 - 3x$

KE

$A = x(12-3x) = 12x - 3x^2$

$\frac{dA}{dx} = 12 - 6x \qquad \frac{dA}{dx} = 0 \Rightarrow x = 2$

$y = 6$