Duality for non-compact manifolds

Thm $D_M : H^k_c(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k if M is an R-oriented n-manifold.

Note: if M compact $H^k_c(M;R) = H^k(M;R)$.

Q: what is D_M? $\underbrace{K}_{\text{compact}} \subset L \subset M$. **recall** $H_n(M|K) = H_n(M, M-K)$.

$$\begin{array}{ccc} H_n(M|L;R) \times H^k(M|L;R) & \xrightarrow{\frown} & \\ \downarrow i_* \qquad\qquad \uparrow i^* & & H_{n-k}(M;R) \\ H_n(M|K;R) \times H^k(M|K;R) & \xrightarrow{\frown} & \end{array}$$

M R-orientable means we have orientation classes $\mu_K \in H_n(M|K;R)$, $\mu_L \in H_n(M|L;R)$

s.t. $i_*(\mu_L) = \mu_K$.

cap product natural: $i_*(\mu_L) \frown x = \mu_L \frown i^*(x)$ $\quad x \in H^k(M|K;R)$.

so $\mu_K \frown x = \mu_L \frown i^*(x)$

Let K vary over compact sets in M, & then $H^k(M|K;R) \to H_{n-k}(M;R)$

get direct limit homomorphism

$$D_M : H^k_c(M;R) \to H_{n-k}(M;R)$$

Proof (Poincaré duality) (sketch)

(A) $M = U \cup V$ union of two open sets, so if $D_U, D_V, D_{U\cap V}$ then Mayer-Vietoris thm and five lemma $\Rightarrow$ D_M isomorphism.

MV:

$$\begin{array}{ccccccccc} \cdots \to & H^k_c(U\cap V) & \to & H^k_c(U) \oplus H^k_c(V) & \to & H^k_c(M) & \to & H^{k+1}_c(U\cap V) & \to \cdots \\ & \downarrow D_{U\cap V} & & \downarrow D_U \oplus D_V & & \downarrow D_M & & \downarrow D_{U\cap V} & \\ \cdots \to & H_{n-k}(U\cap V) & \to & H_{n-k}(U) \oplus H_{n-k}(V) & \to & H_{n-k}(M) & \to & H_{n-k-1}(U\cap V) & \to \cdots \end{array}$$

commutes up to sign.

(B) Let $M = U_1 \subset U_2 \subset U_3 \subset \cdots$ increasing union of open sets and each $D_{U_i} : H^k_c(U_i) \to H_{n-k}(U_i)$ is an isomorphism, then so is D_M by taking direct limits.

① do $\mathbb{R}^n$ = interior of Δ^n, where D_M is the map $H^k(\Delta^n, \partial\Delta^n) \to H_{n-k}(\Delta^n)$

② do $\underset{\text{open}}{U} \subseteq \mathbb{R}^n$, then U is a union of convex open sets (e.g. balls)

③ now do $X = \bigcup U_i$ open cover. □.

Corollary Let M^n be a closed manifold with n odd. Then $\chi(M^n) = 0$.

Proof ① M orientable, then rank $H_i(M;\mathbb{Z})$ = rank $H^{n-i}(M;\mathbb{Z})$ = rank$(H_{n-i}(M;\mathbb{Z}))$ by universal coefficients, so all terms in $\sum_{i=0}^{n}(-1)^i$ rank $H_i(M;\mathbb{Z})$ cancel in pairs.

② M non-orientable replace $H_i(M;\mathbb{Z})$ by $H_i(M;\mathbb{Z}_2)$.

Fact $\chi(M) = \sum_i (-1)^i \operatorname{rank}(H_i(M;\mathbb{Z})) = \sum_i (-1)^i \operatorname{rank}(H_i(M;\mathbb{Z}_2))$.

universal coeffs ~~Poincaré duality~~ : $H_i(M;\mathbb{Z}_2) \cong H^i(M;\mathbb{Z}_2)$ ← as field coeffs.

→ $\mathbb{Z}$ ~~factors~~ summands of $H_i(M;\mathbb{Z})$ give $\mathbb{Z}_2$ summands of $H^i(M;\mathbb{Z}_2)$.

$\mathbb{Z}_m$ summands of $H_i(M;\mathbb{Z})$ — m odd – goes away

m even gives $\mathbb{Z}_2$ summand of $H^i(M;\mathbb{Z}_2)$ and $H^{i-1}(M;\mathbb{Z}_2)$ □

Nonsingularity

Prop$^{\underline{n}}$ $\psi(\alpha \frown \phi) = (\phi \smile \psi)(\alpha)$ $\alpha \in C_{k+\ell}(X;R)$, $\phi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$.

Proof $\psi(\sigma \frown \phi) = \psi(\phi(\sigma|[v_0,\ldots,v_k])\,\sigma|[v_k,\ldots,v_{k+\ell}])$

$= \phi(\sigma|[v_0,\ldots,v_k])\,\psi(\sigma|[v_k,\ldots,v_{k+\ell}]) = (\phi \smile \psi)(\sigma)$. □

so

$$C^\ell(X;R) \xrightarrow{\phi\cup\cdot} C^{k+\ell}(X;R) \rightsquigarrow H^{\ell}(X;R) \to H^{k+\ell}(X;R).$$
$$C_\ell(X;R) \xleftarrow{\cdot\cap\phi} C_{k+\ell}(X;R) \rightsquigarrow \mathrm{Hom}(H_\ell(X;R),R) \to \mathrm{Hom}(H_{k+\ell}(X;R),R).$$

$$\begin{array}{ccc} H^\ell(X;R) & \xrightarrow{h} & \mathrm{Hom}(H_\ell(X;R),R) \\ \downarrow \phi\cup & & \downarrow (\cap\phi)^* \\ H^{k+\ell}(X;R) & \xrightarrow{h} & \mathrm{Hom}(H_{k+\ell}(X;R),R) \end{array}$$

__Fact__ h maps are isos if
- R field
- $\mathbb{Z}$ no torsion.

} (*)

if (*) holds then $\phi\cup$ is the dual of $\cap\phi$. In particular:

consider:

$$H^k(M;R)\times H^{n-k}(M;R) \longrightarrow R$$
$$(\phi, \psi) \longmapsto (\phi\cup\psi)[M].$$

__Thm__ The cup product pairing is non-singular for closed R-orientable manifolds when R is a field, or when $R=\mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.

__Proof__ $$H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R),R) \xrightarrow{D^*} \mathrm{Hom}_R(H^k(M;R),R)$$

h ↗ map from universal coefficients

D^* ↑ hom dual of Poincaré duality map $D: H^k \to H_{n-k}$.

$$D^*h: \psi \longmapsto \phi \mapsto \psi([M]\cap\phi)$$
$$= (\phi\cup\psi)[M]$$

D iso, h iso $\Rightarrow$ D^*h iso. $\Rightarrow$ non-singular in one variable.

$\phi\cup\psi = \pm\psi\cup\phi \Rightarrow$ non-singular in other variable □.

__Def__ A bilinear map $A\times B\to R$ is __non-singular__ if the maps $A\to \mathrm{Hom}(B,R)$ and $B\to\mathrm{Hom}(A,R)$ are both isomorphisms.

Corollary M closed orientable n-manifold, let $\alpha \in H^k(M;\mathbb{Z})$, infinite order, primitive, then there is $\beta \in H^{n-k}(M;\mathbb{Z})$ s.t. $\alpha \cup \beta$ generates $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$
with field coeffs, this holds for any $\alpha \neq 0$.

Proof α generates a $\mathbb{Z}$-summand of $H^k(M;\mathbb{Z}) \cong \langle\alpha\rangle \oplus G$.
there is a homomorphism $\phi: H^k(M;\mathbb{Z}) \to \mathbb{Z}$
$\alpha \mapsto 1$

$$\phi = \alpha \cup \beta \text{ for some } \beta \in H^{n-k}(M;\mathbb{Z}), \text{ i.e. } (\alpha \cup \beta)[M] = 1$$

So $\alpha \cup \beta$ generates $H^n(M;\mathbb{Z})$ □.

Example ① $\mathbb{C}P^n$ ← orientable $2n$-manifold $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$
$|\alpha| = 2$.

$\mathbb{C}P^2$: H^*:

	0	1	2	3	4	
	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	
	1		α		$\alpha \cup \alpha$	generates.

$\mathbb{C}P^{n-1} \overset{i}{\hookrightarrow} \mathbb{C}P^n$ gives ring isomorphism on $H^k(\,;\mathbb{Z})$ for $k \leq 2n-1$
but then by induction $H^2(\mathbb{C}P^n;\mathbb{Z}) \times H^{2n-2}(\mathbb{C}P^n;\mathbb{Z}) \to H^{2n}(\mathbb{C}P^n;\mathbb{Z})$
$\alpha \cup \alpha^{n-1}$
α^n generates.

② $\mathbb{R}P^n$ w/ $\mathbb{Z}_2$ coeffs. similar $\mathbb{R}P^2$: H^*

0	1	2	
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
1	α	$\alpha \cup \alpha$	generates.

Other versions of duality Thm
Poincaré–Lefschetz duality for manifolds with boundary: let $\partial M = A \cup B$ $\partial A = \partial B = A \cap B$
then there is an isomorphism $D_M: H^k(M, A; R) \xrightarrow{\sim} H_{n-k}(M, B; R)$ $\forall k$
$\phi \longmapsto [M] \cap \phi$.

Thm Alexander duality K compact, locally contractible, proper subspace of S^n
then $\tilde{H}_i(S^n - K;\mathbb{Z}) \cong \tilde{H}^{n-i-1}(K;\mathbb{Z})$ $\forall i$