Q: can we describe $H^i_c(X;G)$ in terms of $H^i(X, X-K; G)$?
A: yes, use direct limits.

Defn Observation $C^i_c(X;G) = \bigcup_K C^i(X, X-K; G)$

and each inclusion $K \hookrightarrow L$ induces inclusions $C^i(X,X-K;G) \hookrightarrow C^i(X, X-L; G)$
which induce maps $H^i(X,X-K;G) \to H^i(X,X-L;G)$ ← problem not nec. injective!
but can still describe $H^i_c(X;G)$ as something to do with $H^i(X,X-K;G)$

Defn A directed set I is a partially ordered set s.t. for all $\alpha, \beta \in I$, there is $\gamma \in I$ s.t. $\alpha \le \gamma$ and $\beta \le \gamma$.

Defn A collection of abelian groups $\{G_\alpha \mid \alpha \in I\}$ is called a directed system of groups if I is a directed set, and for all $\alpha \le \beta$ there are homomorphisms $f_{\alpha\beta}: G_\alpha \to G_\beta$ s.t. $f_{\alpha\alpha} = \mathbb{1}_{G_\alpha}$ and if $\alpha \le \beta \le \gamma$

$$\begin{tikzcd} G_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[rr, bend right, "f_{\alpha\gamma}"'] & G_\beta \arrow[r, "f_{\beta\gamma}"] & G_\gamma \end{tikzcd}$$

$f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$

Defn direct limit $\varinjlim G_\alpha$

① $\bigoplus_\alpha G_\alpha / H$ $\quad H =$ subgroup generated by all elements of the form $a - f_{\alpha\beta}(a)$ $\quad (\ldots 0, \ldots, a, \ldots, -f_{\alpha\beta}(a), \ldots)$

equivalently
② $\bigsqcup_\alpha G_\alpha / \sim$ equivalence relation $a \sim b$ if $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$

for some γ, where $a \in G_\alpha$, $b \in G_\beta$.

Useful properties

1) if I has a maximal element γ, then $\varinjlim G_\alpha = G_\gamma$
2) if $J \subset I$ has property that for any $\alpha \in I$, $\exists \beta \in J$ with $\alpha \le \beta$ then $\varinjlim_I G_\alpha = \varinjlim_J G_\alpha$.

Examples ① $I = \mathbb{N}$, $a \le b$ iff $a \mid b$ $\quad G_a = \mathbb{Z}$ for all a, define $f_{ab}: \mathbb{Z} \xrightarrow{\times b/a} \mathbb{Z}$

then $\begin{tikzcd} G_a \arrow[r, "\times b/a"] \arrow[rr, bend right, "\times c/a"'] & G_b \arrow[r, "\times c/b"] & G_c \end{tikzcd}$ then $\varinjlim_{\mathbb{N}} G_\alpha = \mathbb{Q}$.

② $I = \mathbb{N}$ $a \leq b$ iff $a \mid b$ $G_a = \mathbb{Z}_a$ and define $f_{ab} : \mathbb{Z}_a \xrightarrow[\times b/a]{} \mathbb{Z}_b$ give $\mathbb{Q}/\mathbb{Z}$.

<u>Prop^n</u> $H^i_c(X;G) = \varinjlim H^i(X, X-K; G)$ where directed set is compact subsets of X ~~order~~ partially ordered by inclusion.

<u>Proof</u> consider $C^i_c(X;G) = \bigcup_K C^i(X, X-K; G)$.

any cochain (cycle) in $C^i_c(X;G)$ is contained in some $C^i(X, X-K; G)$, and if a cocycle is zero in $C^i_c(X;G)$ it is zero in $C^i(X, X-L; G)$ for some $K \subseteq L$ $\square$

<u>Observation</u> could replace directed set by only considering a collection of compact sets K_α s.t. every compact set $K \subseteq K_\alpha$ for some α.

<u>Example</u> $H^*_c(\mathbb{R}^n; G) = \varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n - K; G) = \varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n - B_r; G)$
where B_r = ball of radius r at origin (can take $r \in \mathbb{N}$).
recall: $H^i(\mathbb{R}^n, \mathbb{R}^n - B_r; G) = \begin{cases} 0 & \text{unless } i = n \\ G & \text{if } i = n \end{cases}$

and the maps $H^i(\mathbb{R}^n, \mathbb{R}^n - B_r; G) \to H^i(\mathbb{R}^n, \mathbb{R}^n - B_{r+1}; G)$ isomorphisms.

$\Rightarrow H^i_c(\mathbb{R}^n; G) = \begin{cases} 0 & i \neq n \\ G & i = n \end{cases}$

<u>Warning</u> $H^i_c(X;G)$ <u>not</u> an invariant of homotopy type.

<u>Problem</u> $f: X \to Y$ in general does not induce a map $f^*: H_c(Y) \to H_c(X)$

e.g. $f: \mathbb{R}^n \to \{pt\}$.

$\downarrow \quad \downarrow$

$\underbrace{G}_{\text{not compactly supported.}} \quad \underbrace{G}_{\text{compactly suppor}}$

<u>but</u> works for <u>proper</u> maps $f: X \to Y$ s.t. $f^{-1}(\text{compact})$ is compact.