<u>Recall</u> M^n orientable, $H_n(M^n;\mathbb{Z}) \cong \mathbb{Z}$ ← generator one copy of every n-simplex with appropriate sign.

M^n not orientable $H_n(M^n;\mathbb{Z}) \cong 0$

$H_n(M^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$ ← generator one copy of every simplex

<u>Def$^{\underline{n}}$</u> <u>cap product</u> X topological space, R coefficient ring

$$C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R) \qquad (k \geqslant \ell).$$

$$(\sigma\ ,\ \phi) \longmapsto \sigma \frown \phi = \phi(\sigma|[v_0,\dots,v_\ell])\,\sigma|[v_\ell,\dots,v_k]$$

$\sigma : \Delta^k \to X$, $\phi \in C^\ell(X;R)$.

<u>Prop$^{\underline{n}}$</u> $\partial(\sigma\frown\phi) = (-1)^\ell(\partial\sigma \frown \phi - \sigma \frown \delta\phi)$

<u>Proof</u> $\partial(\sigma\frown\phi) = \displaystyle\sum_{i=\ell}^{k} (-1)^{i-\ell}\phi(\sigma|[v_0,\dots,v_\ell])\,\sigma|[v_\ell,\dots,\hat{v}_i,\dots v_k]$.

$$\partial\sigma\frown\phi = \sum_{i=0}^{\ell}(-1)^i\phi(\sigma|[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}])\,\sigma|[v_{\ell+1},\dots,v_k].$$

$$+ \sum_{i=\ell+1}^{k}(-1)^i\phi(\sigma|[v_0,\dots,v_\ell])\,\sigma|[v_\ell,\dots,\hat{v}_i,\dots,v_k].$$

$$\sigma\frown\delta\phi = \sum_{i=0}^{\ell}(-1)^i\phi(\sigma|[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}])\,\sigma|[v_{\ell+1},\dots,v_k]. \quad \square.$$

<u>Corollary</u>: $\frown : H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R)$.

<u>Proof</u> check: cycle $\frown$ ~~cobar~~ cocycle is cycle.

α $\qquad$ β.

$\partial\alpha = 0 \qquad \delta\beta = 0. \qquad \partial(\alpha\frown\beta) = \pm(\underbrace{\partial\alpha}_{0}\frown\beta - \alpha\frown\underbrace{\delta\beta}_{0})$ ✓

well defined:

boundary $\frown$ cocycle is boundary

$\alpha = \partial\gamma \qquad \beta,\ \delta\beta = 0 \qquad \alpha\frown\beta = \partial\gamma\frown\beta = \pm\partial(\gamma\frown\beta) \pm \gamma\frown\underbrace{\delta\beta}_{0}$ ✓

cycle $\cap$ coboundary is boundary

α $\quad$ β

$\partial\alpha = 0$ $\quad$ $\beta = \delta\sigma$

$$\alpha \cap \beta = \alpha \cap \delta\beta = \pm\, \partial(\alpha \cap \beta) \pm \underbrace{\partial\alpha}_{0.} \cap \beta$$

boundary ✓.

$\square$.

__Fact__: there are relative versions

$$H_k(X,A;R) \times H^l(X;R) \xrightarrow{\cap} H_{k-l}(X,A;R)$$
$$H_k(X,A;R) \times H^l(X,A;R) \xrightarrow{\cap} H_{k-l}(X;R)$$
$$H_k(X,A\cup B;R) \times H^l(X,A;R) \xrightarrow{\cap} H_{k-l}(X,B;R).$$

__Naturality__ $\quad f_*(\alpha) \cap \phi = f_*(\alpha \cap f^*(\phi))$.

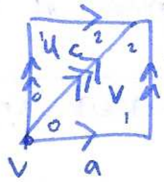
$$\begin{array}{ccccc} H_k(X) & \times & H^l(X) & \xrightarrow{\cap} & H_{k-l}(X) \\ \downarrow f_* & & \uparrow f^* & & \downarrow f_* \\ H_k(Y) & \times & H^l(Y) & \xrightarrow{\cap} & H_{k-l}(Y) \end{array}$$

__Thm__ (Poincaré duality) M closed R-orientable manifold, with fundamental class $[M] \in H_n(M;R)$ then the map $D: H^k(M;R) \to H_{n-k}(M;R)$

$D: \alpha \longmapsto [M] \cap \alpha$

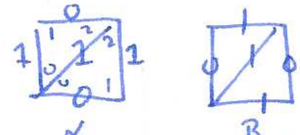
is an isomorphism for all k.

__Example__. $X = T^2$



$[T^2] \in H_2(T^2;\mathbb{Z})$ given by $u - v$

$H^1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2$ generated by

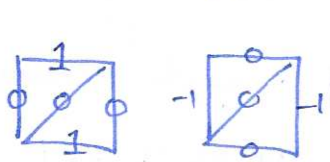


α $\quad$ β.

$D: H^1(T^2;\mathbb{Z}) \to H_1(T^2;\mathbb{Z})$

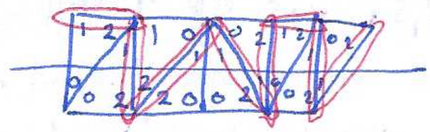
__chain formula__:

$\sigma \cap \phi = \phi(\sigma|[v_0,v_1])\, \sigma|[v_1,v_2]$



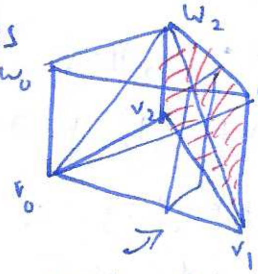
$[M] \cap \alpha = [a]$ $\qquad$ $[M] \cap \beta = -[b]$

more generally: in dim 2:



$[M] \cap H^1 \in H_1$

in dim 3: prisms



2d intersection

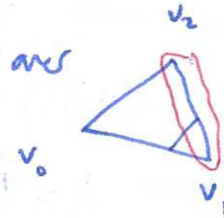
$[M] \cap H^1 \in H_2$

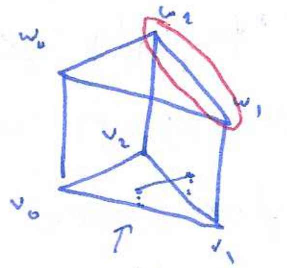
$\underbrace{v_0 w_0}_{0} w_1 w_2$

$\underbrace{v_0 v_1} w_1 w_2$

$\underbrace{v_0 v_1}_{1} v_2 w_2$

gives product over





1d intersection.

$\underbrace{v_0 w_0 w_1}_{0} w_2$

$\underbrace{v_0 v_1 w_1}_{1} w_2$

$\underbrace{v_0 v_1 v_2}_{0} w_2$

Thm (Poincaré duality) $H^k(M^n; R) \cong H_{n-k}(M; R)$

$$\alpha \longmapsto [M] \cap \alpha$$

Proof uses induction + Mayer-Vietoris, will need version of Poincaré duality for open / non-compact subsets of M^n. ← need cohomology with compact supports.

Defn simplicial cohomology with compact supports. X Δ-complex, not nec. finite, but locally compact.

$$\Delta^i_c(X;G) \subset \Delta^i_{\Delta}(X;G)$$

compactly supported cochains on X i.e. non-zero on only finitely many simplices. ↑ cochains on X

note: $\delta(\Delta^i_c(X;G)) \subset \Delta^{i+1}_c(X;G)$, so this gives a subchain complex

$$\cdots \leftarrow \Delta^{i+1}(X;G) \leftarrow \Delta^i(X;G) \leftarrow \cdots$$

$$\cup$$

$$\cdots \leftarrow \Delta^{i+1}_c(X;G) \leftarrow \Delta^i_c(X;G) \leftarrow \cdots$$

← homology groups of this are $H^i_c(X;G)$ (simplicial) cohomology with compact supports.

Example $X=\mathbb{R}\Delta pt$ $\quad v_{-1}\ \ v_0\ \ v_1\ \ v_2\ \ v_3$

so $H^0(\mathbb{R};\mathbb{Z})\cong\mathbb{Z}$
$H^1(\mathbb{R};\mathbb{Z})\cong 0$

$$\begin{array}{ccccccc} 0 & \leftarrow & \Delta^1(X;\mathbb{Z}) & \leftarrow & \Delta^0(X;\mathbb{Z}) & \leftarrow & 0 \\ & & \cup & & \cup & & \\ 0 & \leftarrow & \Delta^1_c(X;\mathbb{Z}) & \leftarrow & \Delta^0_c(X;\mathbb{Z}) & \leftarrow & 0 \end{array}$$

<u>d=0</u>
spse $\phi\in\Delta^0(X;\mathbb{Z})$ and $\delta\phi=0$

$\Rightarrow \quad \delta\phi(\sigma)=\phi(\partial\sigma)=0 \quad \Rightarrow \quad \phi(v_{n+1})-\phi(v_n)=0$

$\Rightarrow \quad \phi(v_{n+1})=\phi(v_n) \quad \Rightarrow \quad \phi=\text{const}$

$\Rightarrow \quad \phi\notin\Delta^0_c(X;\mathbb{Z})$!

so no cocycles in $\Delta^0_c(X;\mathbb{Z})$ except 0. $\Rightarrow \quad H^0_c(\mathbb{R};\mathbb{Z})=0$

<u>d=1</u> consider $\Sigma:\Delta^1_c(\mathbb{R};\mathbb{Z})\to\mathbb{Z}$

~~not defined on all of Δ~~

$\phi\mapsto \sum_{n=-\infty}^{\infty}\phi([v_n,v_{n+1}])$ ← not well defined on $\Delta^1(X;\mathbb{Z})$!
<u>but</u> defined on $\Delta^1_c(X;\mathbb{Z})$.

suppose $\phi=\delta\psi$ then $\phi([v_n,v_{n+1}])=\psi([v_{n+1}])-\psi([v_n])$

so $\quad \Sigma(\phi)=\sum_n \psi([v_{n+1}])-\psi([v_n])=0$.

so $\Sigma: H^1_c(\mathbb{R};\mathbb{Z})\to\mathbb{Z}$ and is surjective, e.g. as $\quad \cdots 0\ 0\ n\ 0\ 0 \cdots$ (vertices $-1\ 0\ 1\ 2\ 3$) $\xrightarrow{\Sigma} n$.

<u>claim</u>: Σ is injective, i.e. if $\Sigma(\phi)=0$ then $\phi=\delta\psi$ $\quad \cdots 0,\ 0,\ 1,\ 0,\ 0,\ -1,\ 0,\ 0 \cdots$ (vertices $-1\ 0\ 1\ 2\ 3\ 4$)

define $\psi(n)=\sum_{i=-\infty}^{n}\phi([n,n-1])$

then $\delta\psi([n,n-1])=\psi(n)-\psi(n-1)=\sum_{i=-\infty}^{n}\phi([i,i-1])-\sum_{i=-\infty}^{n-1}\phi([i,i-1])=\phi(n)$.

<u>summary</u> for $\mathbb{R}$: $\quad H^0(\mathbb{R};\mathbb{Z})\cong\mathbb{Z} \quad H^0_c(\mathbb{R};\mathbb{Z})=0$
$H^1(\mathbb{R};\mathbb{Z})\cong 0 \quad H^1_c(\mathbb{R};\mathbb{Z})\cong\mathbb{Z}$.

<u>version for singular homology</u> <u>Defn</u> $\quad C^i_c(X;G)<C^i(X;G)$

consists of cochains $\phi\in C^i(X;G)$ such that there is a compact set $K_\phi\subset X$
s.t. $\phi=0$ on all chains contained in $X-K_\phi$

note: $\delta\phi=0$ on all $(i+1)$-chains contained in $X-K_\phi$ so $C^i_c(X;G)$ forms a subcomplex of $C^i(X;G)$.