Review UFC-(cohomology).

Thm. C chain complex, homology groups $H_n(C)$, then the cohomology groups $H^n(C_n,G)$ of the dual cochain complex $\mathrm{Hom}(C,G)$ are determined by the split exact sequences:

$$0 \to \mathrm{Ext}(H_{n-1}(C),G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G) \to 0 \quad \square$$

How to compute $\mathrm{Ext}(H,G)$: 1) $\mathrm{Ext}(H\oplus H',G) \cong \mathrm{Ext}(H,G)\oplus \mathrm{Ext}(H',G)$

2) $\mathrm{Ext}(H,G)=0$ if H is free.

3) $\mathrm{Ext}(\mathbb{Z}_n,G)= G/nG$.

Corollary $H^n(C;\mathbb{Z}) \cong (H_n/T_n)\oplus T_{n-1}$

UFC (homology).

Thm. C chain complex, then $0\to H_n(C)\otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C),G)\to 0$ is natural short exact, split but not naturally

Corollary $0\to H_n(X,A)\otimes G \to H_n(X,A;G) \to \mathrm{Tor}(H_{n-1}(X,A),G)\to 0$ split exact; sequence natural wrt $(X,A)\to(Y,B)$ etc.

How to compute $\mathrm{Tor}(A,B)$: 1) $\mathrm{Tor}(A,B)\cong \mathrm{Tor}(B,A)$

2) $\mathrm{Tor}(\bigoplus_i A_i,B) \cong \bigoplus_i \mathrm{Tor}(A_i,B)$.

3) $\mathrm{Tor}(A,B)=0$ if A or B is free (or torsion free)

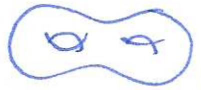
4) $\mathrm{Tor}(A,B)\cong \mathrm{Tor}(T(A),B)$ $T(A)$ torsion subgroup.

5) $\mathrm{Tor}(\mathbb{Z}_n,A)\cong \ker(A\xrightarrow{n}A)$

6) for each short exact sequence $0\to B\to C\to D\to 0$ there is a natural short exact sequence $0\to \mathrm{Tor}(A,B)\to\mathrm{Tor}(A,C)\to\mathrm{Tor}(A,D)\to$

$\to A\otimes B \to A\otimes C \to A\otimes D \to 0$.

If M has an orientation we say M is orientable.

Examples. S^1 , S^2 , , , S^n, ... $\mathbb{RP}^3$! orientable.

Non orientable: Möbius band , $\mathbb{RP}^2$, K, $\mathbb{RP}^n$, n even.

Observation $H_n(M|x)$ has exactly two generators ± 1, this defines a 2-1 covering map $\tilde{M} \to M$. ← local charts are just open sets $x \in \mathring{B}^n \subseteq M^n$.

Claim $\tilde{M}$ is orientable.

Proof for $\mu_x \in \tilde{M}$ choose the local orientation given by $\tilde{\mu}_x \in H_n(\tilde{M}|\mu_x)$ corresponding to μ_x using the isomorphisms $H_n(\tilde{M}|\mu_x) \cong H_n(U(\mu_B)|\mu_x) \cong H_n(B|x)$ ← these satisfy local consistency conditions by construction! □.

Prop^n If M is connected, then M is orientable iff $\tilde{M}$ has two components, in particular: $\pi_1 M$ trivial $\Rightarrow$ M orientable

$\pi_1 M$ no subgp of index $2 \Rightarrow M$ orientable. □.

Proof (sketch) M non-orientable $\Rightarrow$ orientation reversing path loop, i.e. $\gamma: I \to M$

$\Rightarrow \tilde{M}$ connected.

end up with different choices of generator.

in particular: double covers $\Leftrightarrow$ index two subgroups. □.

Def^n R-orientation ← consistent choice of generator of $H_n(M|x; R)$.

Def^n $\mathbb{Z}_2$-orientation ← consistent choice of generator $H_n(M|x; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Prop^n Every manifold is $\mathbb{Z}_2$-orientable □.

Thm M closed connected n-manifold. Then

a) If M is R-orientable then $H_n(M;R) \to H_n(M|x;R)$ is an isomorphism for all $x \in M$.

b) If M is not R-orientable, the map $H_n(M;R) \to H_n(M|x;R)$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.

c) $H_i(M;R) = 0$ for $i > n$.

In particular $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } M \text{ is orientable} \\ 0 \text{ if } M \text{ is non-orientable} \end{cases}$

$$H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2 .$$

Defn An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ is a generator for all x is called a fundamental class for M with coefficients in M.

also known as an orientation class.

Observation If M is a Δ-complex:

- $\mathbb{Z}_2$-fundamental class - just choose 1 copy of every n-simplex.
- $\mathbb{Z}$-fundamental class - start with 1 n-simplex, give it an orientation/ordering of the vertices. Now start extending this over adjacent n-simplices. If this extends over all of M, M is orientable. If not, M is not orientable.

Lemma $A \subset M^n$ compact. Then

a) If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$.

b) $H_i(M|A;R) = 0$ for $i > n$.

Defn Let $M_R \to M$ be a covering space and $A \subseteq M$. A section over A is a cts map $s: A \to M_R$ s.t. $p \circ s = id_A$.

Proof (of Lemma) sections correspond to consistent choices of orientations.

Proof (sketch) (of theorem). Let $M_R \to M$ be a cover, let $\Gamma_R(M)$ be the set of sections $s: M \to M_R$. Note:
- sum of two sections is a section
- scalar multiple of a section is a section.

so $\Gamma_R(M)$ is an R-module. There is a homomorphism $H_n(M;R) \to \Gamma_R(M)$, $\alpha \mapsto (x \mapsto \alpha_x)$ where α_x is the image of α under the map $H_n(M;R) \to H_n(M|x;R)$.

Lemma a) $\Rightarrow$ this homomorphism is an isomorphism.
If M connected, then section determined by value at 1 point $\Big\} \Rightarrow$ Thm a, b).

Corollary If M is a closed connected n-manifold, the torsion subgp of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable, and $\mathbb{Z}_2$ if M is non-orientable.

Proof (use UFC for homology).

(split) short exact.

~~$0 \to H_n(M;\mathbb{Z}) \otimes \mathbb{Z} \to H_n(M,\mathbb{Z}) \to \mathrm{Tor}(H_{n-1}(G)$~~

$$0 \to H_n(M;\mathbb{Z}) \otimes G \to H_n(M;G) \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}),G) \to 0$$

orientable: $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$:

$$0 \to G \to H_n(M;G) \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}),G) \to 0$$

spse $H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}_p$; choose $G = \mathbb{Z}_p$:

$$0 \to \mathbb{Z}_p \to H_n(M;\mathbb{Z}_p) \to \mathbb{Z}_p \to 0$$

$\Rightarrow H_n(M;\mathbb{Z}_p) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$. $\#$ $\underset{\text{Fact}}{H_n(M^n;\mathbb{Z}_p) \cong \mathbb{Z}_p}$.

Proof of fact: can make a cell structure on M^n with exactly one n-cell $\square$.

non-orientable: $H_n(M;G) = 0$ unless $G = \mathbb{Z}_2$ in which case $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$.
$\mathbb{Z}_p$.

So $p \neq 2$.

$$0 \to \underset{0}{H_n(M;\mathbb{Z}) \otimes \mathbb{Z}_p} \to \underset{0}{H_n(M;\mathbb{Z}_p)} \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}_p) \to 0$$

$$\Rightarrow \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}_p) = 0.$$

$p=2$:

$$0 \to \underset{0}{\cancel{\mathbb{Z}_2}} \to \mathbb{Z}_2 \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}_2) \to 0.$$

$$\Rightarrow \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}),\mathbb{Z}_2) \cong \mathbb{Z}_2$$

$$\Rightarrow \mathrm{tors}\ H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}_2. \quad \square.$$

Warning: Prop^n M connected non-compact n-manifold then $H_i(M;R) = 0$ for $i \geq n$.

Proof (sketch) orientation/fundamental class would need infinitely many simplices! $\# \square$.

Duality

Thm (Poincaré duality) If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$ then ~~there is a~~ map $D: H^k(M;R) \to H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \cap \alpha$ is an isomorphism for all k.